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Some solutions for the game Odds & Evens.

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The game

Players A and B are each dealt k cards from a pack containing $n = 2r$ cards, numbered $1, 2, ..., n$. Player A chooses a card from those dealt to him; likewise for player B. They show their choices simultaneously. Player A wins if the sum of the two cards is odd; B wins if the sum is even.

The case $n = 8, k = 3$ is discussed on my website, with solution given there.

We give now a solution for other k values and for any even $n \geq 2k$.

The chance that A wins

Suppose player B's k-card hand contains x odd cards (and $k - x$ even cards). Likewise player A is dealt y odd cards. The probability of this joint event is

$$p_k(x, y) = \frac{\binom{r}{x}\binom{r}{k-x}\binom{r-x}{y}\binom{r-k+x}{k-y}}{\binom{n}{k}\binom{n-k}{k}} \qquad \text{where } r = \frac{n}{2}. \tag{1}$$

The usual conventions that apply to binomial coefficients determine the range of x and y within the largest possible range, $0 \leq x, y \leq k$.

Let $a_k(y)$ be the probability that A plays an even card when dealt a hand comprising y odd cards. Also let $b_k(x)$ be the probability that B plays an even card when dealt a hand comprising x odd cards. Obviously $a_k(0) = b_k(0) = 1$ and $a_k(k) = b_k(k) = 0$.

We can write the value of the game, V_k, namely the probability that A wins the game, as

$$\begin{aligned} V_k = \; & p_k(0,k) + p_k(k,0) + \sum_{x=1}^{k-1}\Big(p_k(x,0)(1-b_k(x)) + p_k(x,k)b_k(x)\Big) \\ & + \sum_{y=1}^{k-1}\Big(p_k(0,y)(1-a_k(y)) + p_k(k,y)a_k(y)\Big) \\ & + \sum_{x=1}^{k-1}\sum_{y=1}^{k-1} p_k(x,y)\Big(a_k(y)(1-b_k(x)) + b_k(x)(1-a_k(y))\Big) \\ = \; & p_k(0,k) + p_k(k,0) + \sum_{x=1}^{k-1} p_k(x,0) + \sum_{y=1}^{k-1} p_k(0,y) \\ & + \sum_{x=1}^{k-1}\sum_{y=1}^{k-1} p_k(x,y)\Big(a_k(y)(1-b_k(x)) + b_k(x)(1-a_k(y))\Big) \end{aligned}$$

So

$$V_k = \sum_{x=1}^{k} p_k(x,0) + \sum_{y=1}^{k} p_k(0,y)$$
$$+ \sum_{x=1}^{k-1}\sum_{y=1}^{k-1} p_k(x,y)\Big(a_k(y) + b_k(x) - 2a_k(y)b_k(x)\Big).$$

A turning point of V_k, with respect to vectors $(a_k(1), a_k(1), ..., a_k(k-1))$ and $(b_k(1), b_k(1), ..., b_k(k-1))$, can be found, lying in the region where $0 \le a_k(y) \le 1, \forall y$ and $0 \le b_k(x) \le 1, \forall x$. We find that this turning point is a saddle-point in the sense of differential calculus – and it is also the minimax saddle-point of the 'Odds and Evens' game.

The saddle-point is the unique solution of the following $2k-2$ linear equations of full rank.

$$\sum_{y=1}^{k-1} p_k(x,y)\Big(1 - 2a_k(y)\Big) = 0 \qquad 1 \le x \le k-1 \tag{2}$$

$$\sum_{x=1}^{k-1} p_k(x,y)\Big(1 - 2b_k(x)\Big) = 0 \qquad 1 \le y \le k-1. \tag{3}$$

Since $p_k(x,y) = p_k(y,x)$, these two systems of equations are identical and so yield equal solutions for the a_k-vector and b_k-vector. We focus therefore on just one of the systems, say (2) – which we rewrite as:

$$2\sum_{y=1}^{k-1} p_k(x,y)a_k(y) = \frac{\binom{r}{x}\binom{r}{k-x}}{\binom{n}{k}} - p_k(x,0) - p_k(x,k), \qquad 1 \le x \le k-1. \tag{4}$$

The tables below give the strategies (which are the solutions of (4) and the same for both players) and the game value V_k, for all $k \le 8$.

k	$y=1$	$y=2$	$y=3$	$y=4$	$y=5$	$y=6$
2	$\frac{1}{2}$					
3	$\frac{1}{r-1}$	$\frac{r-2}{r-1}$				
4	$\frac{r+2}{4(r-1)}$	$\frac{1}{2}$	$\frac{3(r-2)}{4(r-1)}$			
5	$\frac{2}{r-1}$	$\frac{11-6r+r^2}{(r-2)(r-1)}$	$\frac{3(r-3)}{(r-2)(r-1)}$	$\frac{r-3}{r-1}$		
6	$\frac{r+9}{6(r-1)}$	$\frac{2(9-5r+r^2)}{3(r-2)(r-1)}$	$\frac{1}{2}$	$\frac{(r-3)(r+4)}{3(r-2)(r-1)}$	$\frac{5(r-3)}{6(r-1)}$	
7	$\frac{3}{r-1}$	$\frac{22-8r+r^2}{(r-2)(r-1)}$	$\frac{6(17-8r+r^2)}{(r-3)(r-2)(r-1)}$	$\frac{(r-4)(27-8r+r^2)}{(r-3)(r-2)(r-1)}$	$\frac{5(r-4)}{(r-2)(r-1)}$	$\frac{r-4}{r-1}$

k	$y=1$	$y=2$	$y=3$	4	$y=5$	$y=6$	$y=7$
8	$\frac{r+20}{8(r-1)}$	$\frac{3(20-7r+r^2)}{4(r-2)(r-1)}$	$\frac{3(104-34r-r^2+r^3)}{8(r-3)(r-2)(r-1)}$	$\frac{1}{2}$	$\frac{5(r-4)(18-5r+r^2)}{8(r-3)(r-2)(r-1)}$	$\frac{(r-4)(r+13)}{4(r-2)(r-1)}$	$\frac{7(r-4)}{8(r-1)}$

Table 1: Values of $a_k(y)$ for $2 \le k \le 8$ and relevant y. Here $r = n/2$.

k	V_k
1 or 2	$\frac{n}{2(n-1)}$
3 or 4	$\frac{n(11-6n+n^2)}{2(n-3)(n-2)(n-1)}$
5 or 6	$\frac{n(274-225n+85n^2-15n^3+n^4)}{2(n-5)(n-4)(n-3)(n-2)(n-1)}$
7 or 8	$\frac{n(13068-13132n+6769n^2-1960n^3+322n^4-28n^5+n^6)}{2(n-7)(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)}$

Table 2: The chance V_k that A wins.

It is interesting that $V_k = V_{k-1}$ when k is even. An insight into why this is so comes from the $a_k(\cdot)$ values when k is even. It turns out that A's optimal strategy when k is even can be achieved by ignoring the last card dealt to A – and then applying the $(k-1)$-strategic rules to the other cards. Likewise for B.

For example, consider the situation when $k = 8$ and the cards dealt to A have $y = 2$ (i.e. 2O+6E). I assert that he can ignore the 8th card dealt to him. With probability $\frac{1}{4}$ the last card will be O leaving a residual of O+6E – and with probability $\frac{3}{4}$ it will be E leaving 2O+5E. In the former case, his strategy uses $a_7(1)$ whilst in the latter his play is based on $a_7(2)$. Thus, by playing in this 'ignore-the-last-card' way,

$$a_8(2) = \frac{1}{4}\, a_7(1) + \frac{3}{4}\, a_7(2) = \frac{1}{4}\frac{3}{r-1} + \frac{3}{4}\frac{22-8r+r^2}{(r-2)(r-1)} = \frac{3(20-7r+r^2)}{4(r-2)(r-1)}.$$

So we see that the optimal strategy for $k = 8$ is achieved by this reduction of cards to 7. In general, it seems from my computational experience that, when k is even,

$$a_k(y) = \frac{y}{k}\, a_{k-1}(y-1) + \frac{k-y}{k}\, a_{k-1}(y). \tag{5}$$

Therefore, assuming (5) is true, it suffices to solve the problem for k odd; the even strategy and game-value follows.

Another relationship observed is:

$$a_k(y) = 1 - a_k(k-y), \tag{6}$$

true for all k. When k is even, this implies that $a_k(k/2) = \frac{1}{2}$. Given that the case k even has effectively been dealt with, it is the application to the case k odd which is important. In this context, we also note the following.

$$a_k(k-1) = \frac{2r-k-1}{2(r-1)} \qquad \text{odd } k \geq 1 \tag{7}$$

$$a_k(k-2) = \frac{(k-2)(2r-k-1)}{2(r-2)(r-1)} \qquad \text{odd } k \geq 3 \tag{8}$$

$$a_k(k-3) = \frac{(2r-k-1)(6-4k+k^2-(k+1)r+r^2)}{2(r-3)(r-2)(r-1)} \qquad \text{odd } k \geq 3 \tag{9}$$

$$a_k(k-4) = \frac{(k-4)(2r-k-1)(6-3k+k^2-2(k+1)r+2r^2)}{2(r-4)(r-3)(r-2)(r-1)} \qquad \text{odd } k \geq 5. \tag{10}$$

Further computations of special cases to $k = 19$ are consistent with all of these empirically-observed identities. It does not seem likely, however, that a general expression for $a_k(y)$ will emerge from recognising such patterns. Similarly, a general form for V_k will be difficult to find.