Probability problems for a 2nd course
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A. Gambler's ruin/simple random walk

1. Consider a polygon with m sides. We start a 'random walk' at one vertex. We move clockwise with probability $\frac{1}{2}$ or anti-clockwise with probability $\frac{1}{2}$ at each move. We proceed in this way, with independent movements, until all sides of the polygon have been traversed (direction of traversal is not important).

   Let the starting vertex be called '1' with other vertices numbered in clockwise order around the polygon. Let A_j be the event that the random walk ends at vertex j.

   (a) Find $P\{A_j\}$ for $j = 1, 2, \cdots, m$.

   (b) For which value of j is this probability a maximum?

   (c) Let N be the number of moves before the random walk ends. Find $E(N)$.

   [Hint: You may quote without proof any results from the lectures on the 'gambler's ruin problem'. For the case $p = q = \frac{1}{2}$, we proved in lectures that the probability of ruin is $1 - \frac{k}{a}$ and the expected number of plays before ruin is $k(a-k)$.]

2. In this problem we consider a generalisation of the gambler's ruin model. We suppose that the probability of winning \$1 depends on the amount of money that our gambler has. If he has j dollars his probability is p_j, whilst his probability of losing is q_j. Find the probability of ruin if he starts with capital k.

3. Suppose a gambler with initial capital k plays against an infinitely rich adversary and wins two units or loses one unit with probabilities p and q, respectively ($p + q = 1$). What is the gambler's chance of being ruined?

4. Consider a random walk which starts at $X_0 = k$ and stops when X_n reaches $a > k$. When X_n becomes zero, however, it is reflected. This means that $X_{n+1} = 1$ with probability r whenever $X_n = 0$; otherwise $X_{n+1} = 0$. It is possible to prove that eventually the walk will be 'absorbed' at 'state' a. Let N = the duration of the game. Find $E(N)$.

5. A man lives in a city with only one TV station. Each night there is a movie shown. There is, independently, a chance α that the man wants to see the movie. But, the chance that he is home on any given night is θ (independently from night to night and independent of his wish to see the TV movie). Fortunately the man has a video recorder which can record the movie when he is not home. His operating policy is as follows. If he is home and wants to see the movie he watches it; if home but not wishing to see tonight's movie, he watches one that was recorded earlier. If he is out, he records the movie if he wishes to see it, provided he has a spare blank tape (or one containing an already 'watched' movie). Assume a tape has capacity for one movie and that he starts with b blank tapes. Let N = number of nights until he needs to buy another tape. Find $E(N)$.
   [Hint: Can you see a random walk?]

B. Practice at regenerative arguments

1. Find a general expression for $E(X_n)$, the expected number of people who "sequentially fill" one row of a theatre, when there is a taboo on people sharing an armrest. (People arrive in sequence and choose a seat at random from those available subject to the taboo.) The row has n seats and $n+1$ armrests. From this general expression, show that $\frac{E(X_n)}{n} \to \frac{1}{2}(1-e^{-2})$ as $n \to \infty$.

2. The theatre-filling problem can be viewed in another way. Suppose you have many planks of wood each cut to a length just a bit wider than a seat, so that it can rest on 2 neighbouring armrests. What is the expected number of planks that can be placed in one row of $n+1$ armrests if the placement is sequential and at random (i.e. equally likely in all available positions). Planks cannot overlap. There is no work to do in this problem, but we should note that, in chemistry, we might have small molecules B (the planks) which bind to a long molecule A (the row of armrests) at random positions such that each B molecule covers two "binding sites" of A. In this chemical reaction we consider that B is at a very high chemical concentration relative to A and that the reaction is irreversible. So $E(X_n)$ gives the final amount of B molecules which an average A molecule binds after taking into account the "steric hindrance" between B molecules.

3. Consider a chemical reaction in which there is one linear molecule A with n sites ("armrests") and a large number of smaller linear molecules B which bind irreversibly to A sequentially. (The molecule A might be a DNA molecule, whilst B might be a small protein which binds to the DNA). A single B molecule, when bound, occupies 3 consecutive sites on A. Find EX_n, where X_n is the number of B molecules which bind to A. Find $\lim_{n\to\infty} E(X_n)/n$.

4. A theatre has n seats arranged in a circle around the stage. If people "sequentially fill" these seats, with a ban on sharing armrests, find EX_n, the expected number of seats filled.

5. Trials are independently a success with probability p or failure with probability q. Find the expected number of trials until we get k consecutive successes. When $k=2$, find the variance of this number.

6. A time zero we have a polygonal "cell" with k sides. At time 1, the cell "divides" in the following random way: one of the $\binom{k}{2}$ pairs of sides is chosen randomly (equally likely) and a line drawn to connect the mid-points of these two sides. Thus at time 1, we have 2 polygonal cells. At time 2 each of these 2 cells divides independently using the same division rules. At times 3, 4, $\cdots$ all cells divide independently. Let X_{nk} be the number of sides of a randomly sampled polygon immediately after the nth round of division given the starting polygon has k sides. Let $b_{nk} = E(X_{nk})$. Find b_{nk} and show that, as $n \to \infty$, $b_{nk} \to 4$ for any $k \geq 3$. Find $\lim_{n\to\infty} \mathrm{Var}(X_{nk})$.

7. A balanced 6-sided die is thrown repeatedly and independently.
   (a) Find the expected number of throws until the sum of scores exceeds 6. [Hint: Set up a recurrence relation. This can be solved by numerical iteration.]

(b) Find the expected number of throws until the product of scores exceeds 10.

C. Generating functions

1. Suppose that probability generating function Q satisfies the functional equation

$$Q(z) = z \exp\left[\lambda\{Q(z) - 1\}\right]$$

Find the mean and variance of a random variable with pgf Q. [Don't try to solve for Q, unless you know about Borel distributions.]

2. A pair of discrete random variables X and Y have their joint probability distribution summarised by a joint probability generating function. That is,

$$Q(u, v) \equiv E(u^X v^Y) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} P\{X = x,\ Y = y\} u^x v^y$$

(a) Show that X and Y are independent iff

$$E(u^X v^Y) = E(u^X) E(v^Y)$$

(b) Show that

$$E(XY) = \left. \frac{\partial^2 Q(u, v)}{\partial u \partial v} \right|_{u=1,\ v=1}$$

(c) A die is thrown successively. Let $X = \#$ of throws until the first 6 and $Y = \#$ throws until the first 1. Find the joint pgf of X and Y. Find $E(XY)$ by using part (b).

D. Random sums

1. In a game of 'modified billiards', a player at each shot scores 0, 1 or 2 with probabilities p, q and r respectively. Each shot is independent of others. However when he fails to score on a particular shot his turn at the table ends and the opponent takes over.

(a) If $K = \#$ of shots while at the table, find the distribution of K.

(b) Find the p.g.f. of $Z =$ total score achieved during the break.

(c) Find $E(Z)$.

Note for teachers: Many students find a random sum here, but the wrong one! They use $Z = X_1 + X_2 + \ldots + X_K$, where $X_i :=$ score on the i-th shot, and state that $E(Z) = E(X)E(K)$. Here K is not independent of the X variates. There is a useful random sum based on $K - 1$, the number of scoring shots, but many students do not find it initially.

2. In the theory of random sums, we have assumed that N is independent of the X variables. Construct an example where N is not independent of the X's and where $E(S_N) \neq E(X)E(N)$.

3. Consider a random sum S_N where $N \sim$ Poisson(λ) and the component variables $X_1, X_2, \cdots, X_N$ are independent Bernoulli variables with parameter p. Show that S_N and $N - S_N$ are independent random variables. [Hint: Find the joint pgf]

E. Branching processes

1. A man has N children, where each one is a boy with probability α or a girl with probability β, independently ($\alpha+\beta = 1$). If N has pgf $K(z) = pz/(1-qz)$, ($p+q = 1$) find the pgf, G say, of B, the number of boys that he has. Find $E(B)$.

2. Suppose that all the male descendants of the man in question 1 sample their number of sons independently using the same G.

   (i) What condition on α and p implies certain extinction of his family name?

   (ii) Assuming that the conditions in part (i) hold, find the mean number of males (including himself) that will have his name.

3. A branching process starts with k ancestors in generation zero. The probability that an individual has x offspring is $(1 + x)\left(\frac{1}{2}\right)^{x+2}$, $x = 0, 1, 2, \ldots$.

   (a) What is the probability generating function (pgf) of the offspring number?

   (b) What is m, the mean offspring number?

   (c) What is the probability that the population will eventually become extinct if $k = 1$?

   (d) What is the probability of eventual extinction if $k = 2$?

4. [*Note* : Throughout this question you may quote without proof any relevant results that you know concerning branching process.] Suppose that we start with a convex polygon having 4 sides. Two of the sides are chosen at random, each of the $\binom{4}{2}$ choices being equally likely. A line is drawn, connecting the mid-points of these two sides. As a result the polygonal 'cell' is divided into two. Each 'daughter' is a convex polygon. [We might think of these polygons as biological 'cells' undergoing division.]

   The environment is very bad for polygons with 3 or 5 sides and these die. Cells with 4 sides survive and, with these cells, the division process continues using the same random rules described in paragraph 1 applied independently for each surviving cell.

   (a) What is the probability that this population of polygonal cells will become extinct?

   (b) Given that extinction occurs, we can define M as the total number of polygons (of all generations) that ever lived. Find $\mathrm{E}(M)$ and $\mathrm{E}(z^M)$ given extinction.

   For another type of 'biological cell', the division rule is slightly different. The division line joins *opposite* sides of the quadrilateral with probability 2/3 and *adjacent* sides with probability 1/3. As before, 3 and 5 sided cells die.

(c) Find the probability of eventual extinction in this case.

F. Markov chains

Note: *the problems starting at number 7 need a matrix-oriented computer package (such as Matlab or SAS-IML).*

1. Three players A, B and C take part in a series of games. First A plays B. Then the winner plays C. The series continues with the winner of each game playing the one who sat out until one player has won twice consecutively. Each game is independent and ties in any game are not possible. Suppose the probabilities of winning any game are: $P\{A \text{ beats } B\} = p_1$; $P\{B \text{ beats } C\} = p_2$; $P\{C \text{ beats } A\} = p_3$.

   Formulate the series as a Markov chain and calculate each player's chances of winning, if $p_1 = p_2 = p_3 = \frac{1}{2}$.

2. A man is assigned to observe a Markov chain which has transition matrix P. However at each time n there is a probability α that he is distracted and does not record the state. Distractions occur independently of the process and of each other. Show that the resultant observed process is a Markov chain with transition matrix $(1-\alpha)P[I-\alpha P]^{-1}$. Show that the limiting state vector of the observed process is the same as that for the original Markov chain (assuming that both exist).

3. A difficult part (b), made easy by part (a).

   (a) A transition matrix is called 'doubly stochastic' if all of its <u>columns</u> sum to 1. Show that if a Markov chain with a doubly stochastic transition matrix P has a limiting state vector π, then π has each element equal to $1/s$, that is, at equilibrium all states are equally likely.

   (b) Consider a circular path passing through m points. At $k < m$ of these points a man stands. At each time $1, 2, \ldots,$ one man is selected at random and asked to move clockwise to the next point. If this destination point is occupied, however, he is not allowed to move. The state of the system is any one of the $\binom{m}{k}$ occupancy patterns and, clearly, the movements from state to state follow a Markov chain. Find the equilibrium probabilities over the states of this system.

4. Consider the following simple No Claim Discount system operated by an insurance company. A motorist with no claims in the previous year pays \$$c$ as premium for the new year, but \$$2c$ if he has made a claim. His accidents form a Poisson process of rate λ. Assuming that every accident leads to a claim, model this as a two-state Markov chain and find P, P^2, P^3, P^n. Find the equilibrium state vector.
   [Note: You may freely quote the lecture notes on 2-state Markov Chains.]

5. Suppose this motorist in problem 4 adopts the rule that he will always claim if the value X of an accident is $> x_1$ (if he is paying \$$c$ at the time) or $> x_2$ (if he is paying \$$2c$). Assuming the accident values have distribution function G, find the modified P. Evaluate the equilibrium state vector and hence the equilibrium

expected premium paid. Your answers will be functions of λ, c, x_1 and x_2. They will also depend on G. [Note: In principle, the motorist could now find the values of x_1 and x_2 to minimise his equilibrium expected payout. Do not do this now as it is algebraicly messy and would, in practice, be done numerically on a computer.]

6. At time 0, a system is in state 0. At times 1, 2, ... a balanced die is thrown, generating independent scores Y_1, Y_2, ... in $\{1, 2, 3, 4, 5, 6\}$. Define $X_n = \left(\sum_{i=1}^{n} Y_i\right)$ mod 4; that is, X_n is the remainder after the sum of dice throws to that stage is divided by 4. The state of the system at time n is X_n. Clearly there are 4 states, $\{0, 1, 2, 3\}$, and the process at times 0, 1, 2, ... is a Markov chain.

   (a) Write down clearly the transition matrix P.

   (b) What is the initial vector π_0?

   (c) Show that P is doubly stochastic and hence conclude what the equilibrium vector π is.

   (d) What is the equilibrium vector π if the Y_i are independent with <u>any</u> distribution $p_0, p_1, p_2, \ldots$ on $\{0, 1, 2, \ldots\}$ such that $p_y > 0$ for all y?

7. In a certain video game, $n \geq 2$ rifleman stand in a circle. At a given time, each independently decides with probability θ to shoot his clockwise neighbour or with probability $\phi = 1 - \theta$ to shoot his anti-clockwise neighbour. There is a chance α that he will be on target, and if so, that neighbour is eliminated for the next round. The process is repeated for those remaining riflemen until there is either one or zero men left. If the state of the system is simply the number of men standing, formulate the transition matrix of this Markov chain when $n = 4$. Using your matrix-oriented software , and the numerical values $\theta = 0.4$ and $\alpha = 0.8$, calculate the probabilities of absorption into states 0 and 1. Also run your programme for $n = 4$, $\theta = 0.5$ and $\alpha = 1$.

8. Consider a game where a woman randomly draws cards <u>with</u> replacement from a standard deck. The cards K, Q, J and 10 score as 10, the Ace scores as 11, whilst other cards score as their face value. After the first draw, she has an 'initial' score in the range $\{2, 3, \ldots, 11\}$. She continues to draw cards, adding the value of the new card to her accumulated score <u>until</u> her score is ≥ 17. If her score is ≥ 22, it is said that she is "too high".

   There is one other rule: if she is "too high" and if one (or more) of her cards is an Ace (currently scored at 11), she rescores one Ace as 1 and continues to draw cards. This "reducing" rule might be applied more than once in the course of the game if needed. On the other hand, if she is too high and has no reducing possibilities, she is called "bust".

   Formulate this game, which provides an approximate model for the croupier's behaviour in the casino game 'Blackjack', as a Markov chain with state space $\{2, 3, \ldots, 10\} \cup (\{11, 12, \ldots, 21\} \times \{0, 1\}) \cup \{bust\}$, 32 states in all. Here the state $(x, y) \in \{11, 12, \ldots, 21\} \times \{0, 1\}$ means that she has a score of x with y unreduced Aces.

Using the theory of lectures and your computing package, find

(a) the probabilities that the dealer in Blackjack stops with scores of 17, 18, 19, 20, 21;

(b) the probabilities that she ends in "bust";

(c) the probabilities of ending in "bust" giving that the first card has score $2, 3, \ldots, 11$.

(d) the probability that n cards will be drawn before the game stops, $n = 2, 3, 4, 5$.

Note: In the real game, cards are drawn without replacement but from a deck comprising many 52-card packs.

9. Five men stand at the corners of a 5-sided polygon and play a game involving two balls. At time zero, players 1 and 3 each have a ball. At subsequent times 1, 2, ..., any man holding a ball throws it to one of his neighbours. It is equally likely that his throw is clockwise or anti-clockwise. The game ends when any man receives two balls simultaneously.

Define the state of the system as follows. State k, where $k = 1, 2, \ldots, 5$, occurs at time n if player k has both balls after the throws at time n occur. State $j + \frac{9i-i^2}{2}$, where $i = 1, 2, \ldots, 4$ and $j = i+1, i+2, \ldots, 5$, occurs at time n if players i and j have a ball each after the throws at time n occur. Thus at time 0, the state is 7 with the possibility of transition to $2, 11, 12$ and 15 from this initial state.

(a) How many states are there? How many of these are absorbing states?

(b) In class we showed that Markov Chains with m absorbing states have a transition matrix P of the form

$$\left[\begin{array}{c:c} I_m & 0 \\ \hdashline R & Q \end{array}\right].$$

Write down R and Q for this problem.

(c) Write a programme to find the probabilities that each player will be the person who has the balls at game end.

G. Problems with a Hong Kong flavour

1. The number of jewellery shops raided in Hong Kong in a given month is Poisson distributed with mean λ. Each raid involves an amount $\$X$, where $X \sim \text{Gamma}(\alpha, \beta)$. Find the moment generating function of the total value of jewellery thefts in a given month. [*Invented during a period when gangs, mainly from China and armed with weapons similar to those used in China's People's Liberation Army, including hand-grenades, would cross into Hong Kong, rob the jewellery shops and flee to China. Very rich pickings, and no death penalty in HK (unlike in China for the same crime)!*

At least that is how it was prior to the handover. Post-handover, there has been one notable instance where a person was captured, tried, convicted and executed in China for a crime committed in HK – contrary to the constitutional arrangements between China and its autonomous region, Hong Kong]

2. A computer software company sells a copy of its product to 1 person. It knows that a person who gets this software distributes it to j friends with probability p_j. Suppose that

$$G(s) \equiv \sum_{j=0}^{\infty} p_j s^j = 1 - \frac{b}{1-p} + \frac{bs}{1-ps}$$

where $b > 0$, $p > 0$ and $b + p < 1$. Find

(a) the expectation m of the p_j distribution;

(b) conditions on b and p such that this chain of illegal users terminates with certainty;

(c) for the case when termination is certain, the pgf of the total number, M, who obtain the software from this sale.

[Note: Throughout this question, assume that the person lives in a country (or territory on the South China coast) with a very large, effectively infinite, population. Also assume that the j people who receive a copy from a given person do not already have the software.]

3. A girl has a 'stored value' ticket for the Hong Kong MTR (the underground railway or metro), with initial value of \$$m$. Each trip for her is either a \$2 trip with probability α or a \$1 trip with probability β ($\alpha + \beta = 1$). If the last trip before expiry costs more that the residual value of the ticket, the girl pays only the residual value.

(a) Let N be the number of trips that she takes on her \$$m$ ticket. Let $d_m = E(N)$. Find a recurrence relationship for d_m for $m \geq 2$.

(b) Let q_m be the probability that she gets a half-price ride on the last trip of her \$$m$ ticket. Find a recurrence relationship for q_m for $m \geq 2$.

(c) Let T be the total 'true' value of the trips taken with this \$$m$ ticket. Show that $E(T) = q_m + m$.

(d) Let X_i be the value of the ith trip. Find $E(X_i)$. Clearly $T = X_1 + X_2 + \cdots + X_N$. Is the random variable N independent of the X variates? (If not, we cannot assume the formula $E(T) = E(X)E(N)$ from our theory of random sums.)

(e) Using $\alpha = 0.4$, calculate from your recurrence relationships in (a) and (b) the values of d_m and q_m for the $m = 0, 1, 2, \ldots, 6$. Also find $E(T)$ and $E(X)E(N)$ for these values.

(f) By introducing a new quantity $e_m = (2 - \beta)d_m - m$, show from (a) that e_m satisfies the recurrence relationship in (b). Thus show that $e_m = q_m$. What does this mean about the relationship between $E(T)$ and $E(X)E(N)$ for general m and α?