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THE RUIN PROBLEM AND COVER TIMES OF ASYMETRIC RANDOM WALKS AND BROWNIAN MOTIONS

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Abstract

A simple asymmetric random walk on the integers is stopped when its range is of a given length. When and where is it stopped? Analogous questions can be stated for a Brownian motion. Such problems are studied using results for the classical ruin problem, yielding results for the *cover time* and the *range*, both for asymmetric random walks and Brownian motion with drift.

RANDOM WALK; COVER TIMES; BROWNIAN MOTION; GENERATING FUNCTIONS; LAPLACE TRANSFORMS; RANGE; STOPPING TIME; FIRST-PASSAGE TIME; WIENER PROCESS.

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1. Introduction

A simple asymmetric random walk on the integer points $\mathbb{Z}$ is stopped when the range first reaches a given magnitude m. The number of steps until this happens is called the *cover time* following terminology established for the symmetric case in Blom, Holst and Sandell [4, p.130] and Aldous and Fill [2].

Our paper gives the most comprehensive results to date on cover times for the asymmetric case. We provide firstly the joint probability generating function of the cover time and two other related variates, namely the position of the walk when stopped (necessarily an extremum of the walk to that time) and the time at which the other extremum is first visited. Various marginal distributions follow, together with the probability that the walk is at a maximum when stopped.

Our work began independently. The third-named author studied cover times using results for the ruin problem as presented in the first edition of the classical treatise *Feller Volume One* (Feller [10]). The two other authors adopted martingale and

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change-of-measure techniques on the same issues, applying their results also to the limiting case of the random walk, namely Brownian motion with drift (Chong and Cowan [6, 7]. Our joint presentation in this paper, firstly for random walks and then in later parts for Brownian motion, adopts the classical and easily-accessible approach based on Feller.

Our results for Brownian motion *with* drift are also new, though they relate to existing less-general results of Imhof [15], who studied range problems for Brownian motion *without* drift, a topic first discussed by Feller [11]. Known results on the range of Brownian motion are also summarised in the authoritative handbook of Borodin and Salminen [5, p124 and p147]; our results cannot be found there.

Some of our *univariate* results for random walks have in part been preempted by the recent paper of Vallois [16] who used martingale techniques to establish certain marginal distributions, but since our results are more general and derivable with only a *Feller volume 1* background, we consider them worthy of reporting.

To commence our theory, we consider the ruin problem in Section 2. Probability distributions and generating functions connected with the range of an asymmetric random walk are obtained in Section 3. In Section 4 we derive Laplace transforms for random variables connected with the coverage of a Brownian motion, treated as a limiting case of the random walk. Finally in Section 5, explicit expressions for the distribution of the cover time are derived by inverting its Laplace transform.

2. The ruin problem

A simple random walk on the integers starts at the origin and has probabilities $p > 0$ and $q := 1 - p > 0$ for steps to the right or left respectively. The walk is absorbed at the (integer) points $-j < 0$ and $m - j > 0$. Let u_{jn} denote the probability of absorption at $-j$ at step n and introduce the generating function

$$U_j(s) := \sum_{n=0}^{\infty} u_{jn} s^n .$$

The probability of absorption at $-j$ is $u_j = U_j(1)$. Such probabilities, often investigated in connection with the *ruin problem*, have a long and interesting history, see Hald [14, Chapters 20 and 23]. Initially we state a theorem which goes back to works of Lagrange in 1777 and Laplace in 1776; our statement of it follows Feller [10, Section 14.5] or [12, Section XIV.5], where a proof may be found, except that we also adopt an alternative hyperbolic function form.

Theorem 1. *For a simple asymmetric random walk as described above and with*

$$\zeta := \ln\sqrt{\frac{p}{q}}, \quad \eta(s) := \operatorname{arccosh}\frac{1}{s\sqrt{4pq}}, \quad \phi(s) := \arccos\frac{1}{s\sqrt{4pq}} = i\eta(s),$$

we have, for $s \neq 0$, that $\eta(1) = |\zeta|$ and

$$U_j(s) = e^{-j\zeta}\,\frac{\sin[(m-j)\phi(s)]}{\sin[m\phi(s)]} = e^{-j\zeta}\,\frac{\sinh[(m-j)\eta(s)]}{\sinh[m\eta(s)]},$$

$$u_j = e^{-j\zeta}\,\frac{\sinh[(m-j)\zeta]}{\sinh[m\zeta]} = q^j\,\frac{p^{m-j}-q^{m-j}}{p^m-q^m}.$$

Furthermore $U_j(\cdot)$ is a rational function with the difference between the degrees of the numerator and the denominator equal to either 0 or 1.

Remarks. (i) $s \in (0, 1/\sqrt{4pq}]$ implies that $\phi(s)$ is imaginary and $\eta(s)$ is real. For $s > 1/\sqrt{4pq}$, however, the situation is reversed. Any inclination to avoid complex-valued functions would lead to the adoption of the *sinh* form, with argument involving η, when $s \in (0, 1/\sqrt{4pq}]$ and Feller's *sin* form involving ϕ when $s > 1/\sqrt{4pq}$. We incline to the use of the hyperbolic *sinh* form with the function η throughout. Nevertheless we have cause occasionally to examine the function η for $s > 1/\sqrt{4pq}$ whereupon it is imaginary, whilst the alternative, ϕ, is real (see Theorem 3). For brevity in typesetting, we define $\mathbf{s}[x] := \sinh(x)$ and $\mathbf{c}[x] := \cosh(x)$ in the sequel.

(ii) If instead the probability for absorption at $m-j$ at step n is desired (as in Theorem 2), then changes of ζ to $-\zeta$ and j to $m-j$ in the formulae above give the appropriate results. $\triangle$

3. Cover time of a simple random walk

Let S_n be the position of the random walk at time n. Suppose that the simple random walk with $S_0 = 0$, namely $\{0, S_1, S_2, \ldots\}$, is stopped when its range is m, that is, when the number of different integers visited (excluding the origin) is m. Hence it is stopped at step

$$N = \min\{n : \max_{0\le i\le n} S_i - \min_{0\le i\le n} S_i = m\}.$$

The coverage of the walk is an interval consisting of $m+1$ consecutive integers (one being 0) with S_N being the point where the walk stops; S_N is an extremum of the walk to time N. The other end point of the coverage interval, also an extremum, is first visited at some step $M < N$, whence

$$S_M = I_{(S_N<0)}(S_N + m) + I_{(S_N>0)}(S_N - m),$$

using $I_{(\cdot)}$ as the indicator function. For the joint probability generating function of N, M and S_N, we have the following result.

Theorem 2. *With $\eta(s) := \operatorname{arccosh}\frac{1}{s\sqrt{4pq}}$, $\zeta(z) := \ln\sqrt{\frac{p}{q}} + \ln z$, we have, for $z > 0$ and $x, y \neq 0$,*

$$\begin{aligned}\mathbb{E}\big(x^M y^{N-M} z^{S_N}\big) &= \Big[\frac{\mathbf{s}[\frac{m}{2}\{\eta(x)+\zeta(z)\}]\mathbf{s}[\frac{m+1}{2}\{\eta(x)+\zeta(z)\}]}{\mathbf{s}[\frac{1}{2}\{\eta(x)+\zeta(z)\}]\mathbf{s}[m\eta(x)]} \\ &\qquad + \frac{\mathbf{s}[\frac{m}{2}\{\eta(x)-\zeta(z)\}]\mathbf{s}[\frac{m+1}{2}\{\eta(x)-\zeta(z)\}]}{\mathbf{s}[\frac{1}{2}\{\eta(x)-\zeta(z)\}]\mathbf{s}[m\eta(x)]}\Big]\frac{\mathbf{s}[\eta(y)]}{\mathbf{s}[(m+1)\eta(y)]} \\ &= \Big[\frac{\mathbf{s}[(m+1)\eta(x)]\,\mathbf{c}[m\zeta(z)] - \mathbf{s}[m\eta(x)]\,\mathbf{c}[(m+1)\zeta(z)] - \mathbf{s}[\eta(x)]}{\{\mathbf{c}[\eta(x)] - \mathbf{c}[\zeta(z)]\}\mathbf{s}[m\eta(x)]\mathbf{s}[(m+1)\eta(y)]}\Big]\mathbf{s}[\eta(y)].\end{aligned}$$

Proof. For $0 < k \le m$ consider $\mathbb{E}(x^M y^{N-M} z^{S_N} I_{(S_N=k)}) = z^k \mathbb{E}(x^M y^{N-M} I_{(S_N=k)})$. A stop at k means that the walk first visits $k-m$, then goes to k without visiting $k-m-1$. The walk can thus be split into two parts: the first from the origin to $k-m$ without visiting k; the second from $k-m$ to k without visiting $k-m-1$. The numbers of steps in the first and the second parts are M and $N-M$ respectively. The two parts are independent, so it follows from the *sinh* form of Theorem 1 that

$$\mathbb{E}(x^M y^{N-M} I_{(S_N=k)}) = e^{-(m-k)\zeta} \frac{\mathbf{s}[k\eta(x)]}{\mathbf{s}[m\eta(x)]} \cdot e^{m\zeta} \frac{\mathbf{s}[\eta(y)]}{\mathbf{s}[(m+1)\eta(y)]}\,.$$

The case $k < 0$ is considered analogously. Combining the cases leads to

$$\begin{aligned}
(1)\quad \mathbb{E}(x^M y^{N-M} z^{S_N}) &= \sum_{k=1}^{m} (e^{k\zeta(z)} + e^{-k\zeta(z)}) \frac{\mathbf{s}[k\eta(x)]}{\mathbf{s}[m\eta(x)]} \cdot \frac{\mathbf{s}[\eta(y)]}{\mathbf{s}[(m+1)\eta(y)]} \\
&= \frac{\mathbf{s}[\eta(y)]}{\mathbf{s}[m\eta(x)]\mathbf{s}[(m+1)\eta(y)]} \sum_{k=1}^{m} 2\mathbf{c}[k\zeta(z)]\mathbf{s}[k\eta(x)] \\
&= \frac{\mathbf{s}[\eta(y)]}{\mathbf{s}[m\eta(x)]\mathbf{s}[(m+1)\eta(y)]} \sum_{k=1}^{m} \Big\{\mathbf{s}[k\{\eta(x)+\zeta(z)\}] + \mathbf{s}[k\{\eta(x)-\zeta(z)\}]\Big\}.
\end{aligned}$$

Further summation gives the first formula in the assertion. The second formula follows by manipulating hyperbolic functions (Abramowitz and Stegun [1, Section 4.5]). $\square$

Remarks. (i) Feller's *sin* form of Theorem 1 could be used equally well, with ϕ appearing instead of η, but the algebraic combination above would only be convenient by defining an imaginary function in place of $\zeta(\cdot)$.

(ii) By martingale and change-of-measure arguments the joint generating function of (N, S_N) is derived in Chong and Cowan [7, Theorem 4] whilst the univariate generating function of N is obtained in Vallois [16, Theorem 2]. One formula from the change-of-measure technique, is implicit in Theorem 2, namely $\mathbb{E}(x^M y^{N-M} z^{S_N} = \mathbb{E}_0\{(\sqrt{4pq}x)^M (\sqrt{4pq}y)^{N-M} (\sqrt{p/q}z)^{S_N}\}$. Here $\mathbb{E}_0$ is expectation under the simple *symmetric* random walk.

(iii) Note that the factorization of our result into factors depending solely on (x,z) and y shows that (M, S_N) and $N-M$ are independent. This is also clear because $N-M$ is the number of steps for the random walk to visit a point at distance m to the right (resp. left) of a starting point *conditioned on* the points left (resp. right) of the starting point not being visited. $\triangle$

We can derive explicit expressions for various distributions, for example, that of N.

Theorem 3. *For $m \ge 2$, $j > 2$, $\mathbb{P}\{N = j\}$ equals*

$$\sqrt{4pq}^{\,j} \Big[\frac{1}{m(pq)^{(m-1)/2}} \sum_{k=1}^{m-1} (-1)^k \frac{(p^{m/2} - (-1)^k q^{m/2})^2}{1 - \sqrt{4pq}\cos\frac{\pi k}{m}} \cos^{j-1}\frac{\pi k}{m} \sin^2\frac{\pi k}{m}$$
$$- \frac{1}{(m+1)(pq)^{m/2}} \sum_{k=1}^{m} (-1)^k \frac{(p^{(m+1)/2} - (-1)^k q^{(m+1)/2})^2}{1 - \sqrt{4pq}\cos\frac{\pi k}{m+1}} \cos^{j-1}\frac{\pi k}{m+1} \sin^2\frac{\pi k}{m+1} \Big],$$

whilst for $m = 2$, $\mathbb{P}\{N = 2\} = p^2 + q^2$.

Proof. Using Theorem 2, with $x = y$ and $z = 1$, it follows that $\mathbb{E}(y^N)$ equals

$$\frac{\mathbf{s}[\eta(y)]}{\mathbf{c}[\eta(y)] - \mathbf{c}[\zeta]} \left[\frac{\mathbf{c}[m\zeta] - \mathbf{c}[\frac{1}{2}\eta(y)]\,/\mathbf{c}[\frac{2m+1}{2}\eta(y)]}{\mathbf{s}[m\eta(y)]} - \frac{\mathbf{c}[(m+1)\zeta] - \mathbf{c}[\frac{1}{2}\eta(y)]\,/\mathbf{c}[\frac{2m+1}{2}\eta(y)]}{\mathbf{s}[(m+1)\eta(y)]} \right].$$

From the proof of Theorem 2 (where rational functions are involved in finite products within a finite sum), we see that the generating function $G(y) := \mathbb{E}(y^N)$ is a rational function with the degree of the numerator exceeding that of the denominator by at most 2. Also, after cancellation of common factors, the roots of the denominator are single; they must satisfy either $\mathbf{s}[m\eta(y)] = 0$ or $\mathbf{s}[(m+1)\eta(y)] = 0\}$. If the roots are denoted by $y_{i,k}$, $i = 1, 2$, then these equations are $m\eta(y_{1,k}) = i\pi k$ and $(m+1)\eta(y_{2,k}) = i\pi k$ for $k \in \mathbb{Z}$, or, recalling that $\eta(y) := \operatorname{arccosh}(1/(y\sqrt{4pq}))$,

$$\begin{aligned} y_{1,k} &= \frac{1}{\sqrt{4pq}\cos\frac{\pi k}{m}}, \qquad k = 1, 2, \ldots, m-1, \quad k \neq m/2, \\ y_{2,k} &= \frac{1}{\sqrt{4pq}\cos\frac{\pi k}{m+1}}, \quad k = 1, 2, \ldots, m, \quad k \neq (m+1)/2. \end{aligned}$$

Note that the root corresponding to $k = 0$ and the root $y_{1,m} = y_{2,m+1}$ are cancelled by the factor $\mathbf{s}[\eta(y)]$ in the numerator. By partial fractions, (Feller [12, pp. 352–354]), we have

$$G(y) = a + by + cy^2 + \sum_{k=1}^{m-1} \frac{d_{1,k}}{y - y_{1,k}} + \sum_{k=1}^{m} \frac{d_{2,k}}{y - y_{2,k}}$$

where $d_{i,k} = \lim_{y \to y_{i,k}} (y - y_{i,k})\, G(y)$. After some calculations we find that

$$\begin{aligned} d_{1,k} &= \frac{\mathbf{c}[m\zeta] - (-1)^k}{\mathbf{c}[\zeta] - \cos\frac{\pi k}{m}} \cdot \frac{(-1)^{k+1}}{m\sqrt{4pq}} \cdot \frac{\sin^2\frac{\pi k}{m}}{\cos^2\frac{\pi k}{m}}, \\ d_{2,k} &= \frac{\mathbf{c}[(m+1)\zeta] - (-1)^k}{\mathbf{c}[\zeta] - \cos\frac{\pi k}{m+1}} \cdot \frac{(-1)^k}{(m+1)\sqrt{4pq}} \cdot \frac{\sin^2\frac{\pi k}{m+1}}{\cos^2\frac{\pi k}{m+1}}. \end{aligned}$$

As

$$G(y) = a + by + cy^2 - \sum_{k=1}^{m} \frac{d_{1,k}}{y_{1,k}} \sum_{j=0}^{\infty} \frac{y^j}{y_{1,k}^j} - \sum_{k=1}^{m+1} \frac{d_{2,k}}{y_{2,k}} \sum_{j=0}^{\infty} \frac{y^j}{y_{2,k}^j},$$

we have, for $j \geq 3$

$$\mathbb{P}\{N = j\} = \sum_{k=1}^{m} \frac{-d_{1,k}}{y_{1,k}^{j+1}} + \sum_{k=1}^{m+1} \frac{-d_{2,k}}{y_{2,k}^{j+1}},$$

which proves the first assertion after simple substitution of $\mathbf{c}[r\zeta]$ by $(p^r + q^r)/(2(pq)^{r/2})$. Finally, for $m = 2$ we note that $N = 2$ iff the first two steps are both in the same direction. The probability for this is $p^2 + q^2$. $\square$

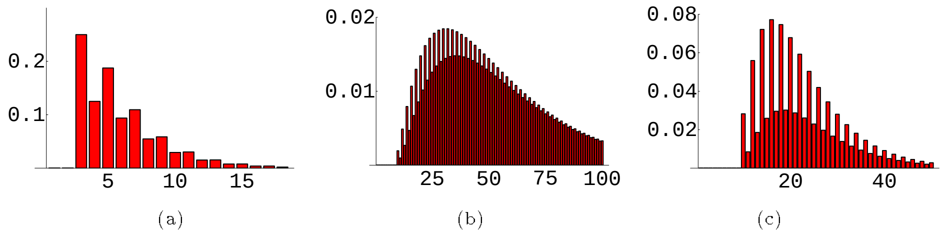


Figure 1: $\mathbb{P}\{N = j\}$ for (a) $p = q = \frac{1}{2}$ and $m = 3$, (b) $p = q = \frac{1}{2}$ and $m = 10$ (where the upper 9% of the pmf is not shown) and (c) $p = 0.7$ and $m = 10$.

Remarks. (i) Figure 1 plots $\mathbb{P}\{N = j\}$ and reveals the interesting phenomenon of alternating magnitudes in the probabilities. This phenomenon is due to simple combinatorics. For example in (a), there are 2 equally-likely sample paths (of probability 1/16 each) which yield $N = 4$ but 6 paths (of probability 1/32 each) which yield $N = 5$. There are more than twice as many paths of odd-length $2k + 1$ to achieve $m = 3$ than there are paths of even length $2k$, more than compensating for the fact that each path of length $2k + 1$ in (a) has half the probability mass of a length-$2k$ path. So probabilities increase from $j = 2k$ to $j = 2k + 1$. In the asymmetric case, there are similar combinatoric aspects.

(ii) In Vallois [16, Theorem 12] an expression for $\mathbb{P}\{N = j\}$ is derived, but in a different way.

(iii) The inequalities in formula (1.6) of Vallois [16] should be strict and this error renders their result in Proposition 14 concerning the distribution function of R_n incorrect. The error is perpetuated in a later paper, Vallois and Tapiero [17].

(iv) Let the range until time n be denoted by $R_n := \max_{0\le j\le n} S_j - \min_{0\le j\le n} S_j$. Using the formula above for $\mathbb{P}\{N = j\}$ and the relation $\mathbb{P}\{R_n < m\} = \mathbb{P}\{N > n\}$, one can derive an explicit expression, valid for general p, for the distribution of R_n (Figure 2). Except for the inaccurate results mentioned in (iii), we know of no published results for the distribution of range R_n, even when $p = q = \frac{1}{2}$, except those of an asymptotic nature as $n \to \infty$, like in Feller [11]. Even the best results to date about the first two moments, namely Feller's formulae for the $p = q = \frac{1}{2}$ case where he shows that $\mathbb{E}(R_n) \sim 2\sqrt{2n/\pi}$ and $\mathbb{V}\mathrm{ar}(R_n) \sim 4n(\ln 2 - 2/\pi)$, are asymptotic in nature.

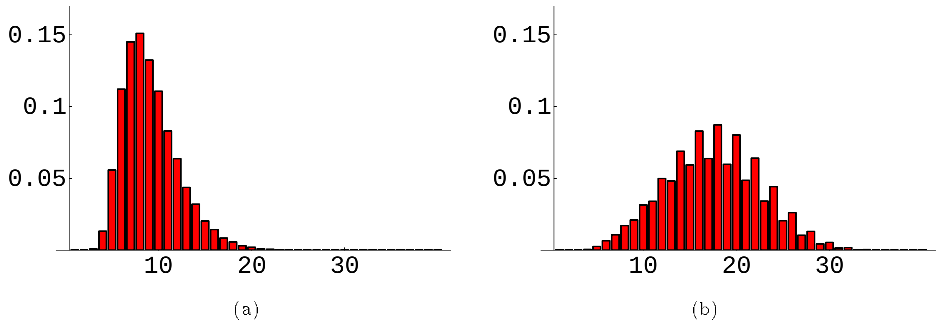


Figure 2: $\mathbb{P}\{R_n = r\}$ for $n = 40$ with (a) $p = q = \frac{1}{2}$ and (b) $p = 0.7$. We note the alternating magnitudes when $p \neq q$, for similar combinatoric reasons as in Figure 1.

(v) Consider a random walk on the vertices of an m-sided polygon, stopped when the walk has traversed all edges of the polygon, thereby completing a *tour* of the circuit. Obviously, in this equivalent formulation, the tour's duration, N, has the distribution given in Theorem 3. $\triangle$

Next consider the coverage of the walk when it is stopped, that is, the interval with end points S_M and S_N.

Theorem 4. *Let* $\zeta := \ln\sqrt{\frac{p}{q}}$. *The distribution of the stopping point is, for* $p \neq q$,

$$\begin{aligned}\mathbb{P}\{S_N = k\} =& e^{k\zeta}\,\frac{\mathbf{s}[|k|\zeta]\mathbf{s}[\zeta]}{\mathbf{s}[m\zeta]\mathbf{s}[(m+1)\zeta]}\\ =& p^{m+(k-|k|)/2}q^{m-(k+|k|)/2}\frac{(p^{|k|}-q^{|k|})(p-q)}{(p^m-q^m)(p^{m+1}-q^{m+1})},\end{aligned}$$

whilst, for $p = q = \frac{1}{2}$, $\mathbb{P}\{S_N = k\} = |k|/[m(m+1)]$. *Here* $k = -m, 1-m, \ldots, m-1, m$.

Proof. Formula (1), with $x = y = 1$, gives the probability generating function of S_N in a form which can be simply reorganised to display the coefficient of z^k. That coefficient is immediately of the form required for $\mathbb{P}\{S_N = k\}$ in this theorem. $\square$

Remarks. (i) The distribution of the maximum of the walk up to stopping is simply, for $\ell = 0, 1, \ldots, m$,

$$\begin{aligned}\mathbb{P}\{\max_{0\le j\le N} S_j = \ell\} =& \mathbb{P}\{S_N = \ell\} + \mathbb{P}\{S_N = \ell - m\}\\ =& \frac{e^{-m\zeta}\mathbf{s}[\zeta]}{\mathbf{s}[(m+1)\zeta]}\cdot e^{2\ell\zeta} = p^\ell q^{m-\ell}\frac{p-q}{p^{m+1}-q^{m+1}} \qquad p \neq q,\\ =& \frac{1}{m+1} \qquad p = q = \tfrac{1}{2}.\end{aligned}$$

(ii) The probability, π_m say, that the random walk is at a maximum when the range of m is first attained, is merely $\mathbb{P}\{S_N > 0\}$ (evaluated later).

(iii) S_M is completely determined by S_N. It is easy to show that, when $p \neq q$,

$$\mathbb{P}\{S_M = 0\} = \frac{2\mathbf{c}[m\zeta]\mathbf{s}[\zeta]}{\mathbf{s}[(m+1)\zeta]} = \frac{(p-q)(p^m+q^m)}{(p^{m+1}-q^{m+1})},$$

whilst

$$\begin{aligned}\mathbb{P}\{S_M = k\} =& \frac{e^{k\zeta(1-m/|k|)}\mathbf{s}[(m-|k|)\zeta]\mathbf{s}[\zeta]}{\mathbf{s}[m\zeta]\mathbf{s}[(m+1)\zeta]}\\ =& [p^k q^m I_{(k>0)} + \frac{p^m}{q^k} I_{(k<0)}]\frac{(p-q)(p^{m-|k|}-q^{m-|k|})}{(p^m-q^m)(p^{m+1}-q^{m+1})}\end{aligned}$$

for $1 \le |k| \le m-1$. When $p = q = 1/2$, $\mathbb{P}\{S_M = k\} = (m-|k|)/(m(m+1))$ for $1 \le |k| \le m-1$, whilst $\mathbb{P}\{S_M = 0\} = 2/(m+1)$.

(iv) When $p = q = 1/2$, the possible $m+1$ values of the midpoint of the range, $\frac{1}{2}(S_M + S_N)$, are equally likely, as shown by Blom, Holst and Sandell [4, p. 131]. $\triangle$

Moments of the random variables M, N, S_N can be simply obtained by differentiation of the joint generating function, but also by the following argument.

Theorem 5. *Let $\zeta := \ln\sqrt{\frac{p}{q}}$. Then for $p \neq q$,*

$$\begin{aligned}
\mathbb{E}(N) &= \frac{1}{p-q}\mathbb{E}(S_N) \\
&= \frac{1}{2(p-q)}\Big[m\coth m\zeta + (m+1)\coth(m+1)\zeta - \coth\zeta - \frac{m(m+1)\mathbf{s}[\zeta]}{\mathbf{s}[m\zeta]\mathbf{s}[(m+1)\zeta]}\Big] \\
&= \frac{m(p-q)(p^m-q^m)(p^{m+1}+q^{m+1}) - m^2(p-q)^2(pq)^m - pq(p^m-q^m)^2}{(p-q)^2(p^m-q^m)(p^{m+1}-q^{m+1})},
\end{aligned}$$

whilst, for $p = q = \frac{1}{2}$, $\mathbb{E}(N) = \frac{1}{2}m(m+1)$ and $\mathbb{E}(S_N) = 0$.

Proof. As N is a stopping time, with S_N and the steps $X_i := S_i - S_{i-1}$ being bounded, we have

$$\mathbb{E}(S_N) = \mathbb{E}\Big(\sum_{i=1}^{N} X_i\Big) = (p-q)\mathbb{E}(N). \tag{2}$$

From this and the explicit expression for $\mathbb{P}\{S_N = k\}$ in Theorem 4, the formula for $\mathbb{E}(N)$ when $p \neq q$ follows after an elementary calculation involving hyperbolic functions.

For $p = q = 1/2$, it is well known that in the classical ruin problem with initial capitals a and b the average duration of the game is ab. Now the cover time N equals

$$1 + \sum_{i=2}^{m}(\text{ the time needed to convert the range from } i-1 \text{ to } i).$$

Each of these conversions is like the duration of a gambler's ruin game, so $\mathbb{E}(N) = 1 + 2\cdot 1 + 3\cdot 1 + \cdots + m\cdot 1 = m(m+1)/2$, proving the second assertion (see Blom, Holst and Sandell [4, p. 130]). $\mathbb{E}(S_N) = 0$ from symmetry considerations or (2). $\square$

Remarks. (i) A lengthy expression for $\mathbb{E}(S_N^2)$ in terms of hyperbolic functions is available in Chong and Cowan [7], where a simple form in terms of p, q and m is also given, as follows. For non-negative integers k, let $g_k := p^k + 2p^{k-1}q + \cdots + (k+1)q^k$, $h_k := (k+1)p^k + kp^{k-1}q + \cdots + q^k$ and $s_k := p^k + p^{k-1}q + \cdots + q^k$. Then

$$\mathbb{E}(S_N^2) = \frac{m(m+1)}{2} + \frac{(g_{m-1} - h_{m-1})(g_m - h_m)}{2s_{m-1}s_m},$$

which equals $\mathbb{E}(N)$ when $p = q = \frac{1}{2}$. In addition they write $\mathbb{E}(S_N)$ in a similar way by showing that

$$\mathbb{E}(N) = \frac{g_{m-1}h_{m-1}}{s_{m-1}s_m}.$$

(ii) They also express π_m, using these same entities, as

$$\pi_m = \frac{p^m g_{m-1}}{s_{m-1}s_m}.$$

(iii) Let $p > q > 0$ be fixed and $m \to \infty$. From the explicit formula for $\mathbb{E}(N)$ one readily obtains

$$\mathbb{E}(N) = \frac{1}{p-q}\left(m - \frac{q}{p-q} + O(m^2(q/p)^m)\right).$$

Using the generating function of N one can show that, $\mathbb{V}\mathrm{ar}(N)/m \to 4pq/(p-q)^3$, and also convergence to the standard normal distribution, $(N - \mathbb{E}(N))/\sqrt{\mathbb{V}\mathrm{ar}(N)} \xrightarrow{d} N(0,1)$, (see Vallois [16, Theorem 20]). These results could be expected from renewal theory for random variables with mean $p - q > 0$ and variance $4pq > 0$ as in our generalised case. If $q > p > 0$ then p and q are interchanged in the above formulae.

(iv) Numerical results for the exact distribution of N indicate that a normal approximation is crude unless $|p - q|$ is quite different from 0 and 1 and m very large. Approximations based on results obtained below for Brownian motions might be more appropriate unless p is close to 0 or 1. $\triangle$

4. Laplace transforms of cover times for Brownian motions

Let the length of the steps in a simple random walk be c and the time between the steps be Δ. Thus, $1/\Delta$ steps of length c are taken each unit of time. For some $\sigma > 0$ and μ, let

$$c^2 = \sigma^2\Delta, \quad p = \frac{1}{2} + \frac{c\mu}{2\sigma^2}, \quad q = \frac{1}{2} - \frac{c\mu}{2\sigma^2}.$$

This gives per unit time a displacement with mean $c(p-q)/\Delta = \mu$ and variance $c^2 4pq/\Delta = \sigma^2 - \mu^2\Delta$. The following lemma is easily proved.

Lemma 6. *For fixed $x \geq 0$, v and $r = mc > 0$ we have as $m \to \infty$,*

$$\begin{aligned} m\zeta &= m\ln\sqrt{p/q} && \to \varepsilon := \frac{r\mu}{\sigma^2}, \\ m\eta(e^{-x\Delta}) &= m\ln\frac{1+\sqrt{1-4pqe^{-2x\Delta}}}{\sqrt{4pqe^{-2x\Delta}}} && \to \alpha(x) := \frac{r}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2} + 2x}\,, \\ m\zeta(e^{-vc}) &= m\zeta - mvc && \to \varepsilon - rv. \end{aligned}$$

Donsker's theorem shows the convergence in distribution,

$$\{cS_{[t/\Delta]} : t \geq 0\} \xrightarrow{d} \{W_t = \mu t + \sigma B_t : t \geq 0\}, \quad c \to 0, \tag{3}$$

where $\{B_t : t \geq 0\}$ is a standard Brownian motion (see Billingsley [3]). The Wiener process $\{W_t : t \geq 0\}$, also called Brownian motion, has drift μ and variance σ^2. Let it be stopped when its range is $r > 0$. The stopping time, that is, the *cover time*, is

$$T = \inf\{t : \sup_{0\leq s\leq t} W_s - \inf_{0\leq s\leq t} W_s = r\}.$$

At that time the coverage of the process is an interval which has end points W_T and W_U, where $U = \inf\{t : W_t = (W_T - r)I_{(W_T>0)} + (W_T + r)I_{(W_T<0)}\}$. Thus the end points of the coverage interval are visited at the times U and T. Note that the random time U is not a stopping time.

The joint Laplace transform of U, T and W_T, a result for Brownian motion with drift, is given in the following theorem.

Theorem 7. *For a Brownian motion with drift μ and variance $\sigma^2 > 0$, we have, for $t, u \geq 0$ and all v, with $\mathbf{s^n}[x] := \sinh^n(x)$ and $\alpha(x) = r/\sigma\ \sqrt{\mu^2/\sigma^2 + 2x}$,*

$$\mathbb{E}(e^{-uU-t(T-U)-vW_T}) = \frac{2\alpha(t)}{\mathbf{s}[\alpha(t)]}\Big[\frac{\mathbf{s^2}[\frac{1}{2}(\alpha(u)+\varepsilon-rv)]}{(\alpha(u)+\varepsilon-rv)\mathbf{s}[\alpha(u)]} + \frac{\mathbf{s^2}[\frac{1}{2}(\alpha(u)-\varepsilon+rv)]}{(\alpha(u)-\varepsilon+rv)\mathbf{s}[\alpha(u)]}\Big].$$

Proof. Using Theorem 2 and Lemma 6 we find that $\mathbb{E}(e^{-uM\Delta-t(N-M)\Delta-vcS_N})$ converges to the RHS of the theorem as $m \to \infty$. From this and the weak convergence in (3) the assertion follows. $\square$

Remarks: (i) The random variable $T-U$ and the random vector (U, W_T) are independent, cf. Theorem 2 remark (iii). For $t, u \geq 0$ and all v,

$$\begin{aligned}
\mathbb{E}(e^{-uU}) &= \frac{2\varepsilon}{\mathbf{s}[\varepsilon]}\Big[\frac{\mathbf{s^2}[\frac{1}{2}(\alpha(u)+\varepsilon)]}{(\alpha(u)+\varepsilon)\mathbf{s}[\alpha(u)]} + \frac{\mathbf{s^2}[\frac{1}{2}(\alpha(u)-\varepsilon)]}{(\alpha(u)-\varepsilon)\mathbf{s}[\alpha(u)]}\Big],\\
\mathbb{E}(e^{-t(T-U)}) &= \frac{\mathbf{s}[\varepsilon]}{\varepsilon}\,\frac{\alpha(t)}{\mathbf{s}[\alpha(t)]},\\
\mathbb{E}(e^{-vW_T}) &= \frac{\varepsilon}{\mathbf{s^2}[\varepsilon]}\Big[\frac{\mathbf{c}[2\varepsilon-rv]-1}{2\varepsilon-rv} + \frac{\mathbf{c}[rv]-1}{rv}\Big].
\end{aligned}$$

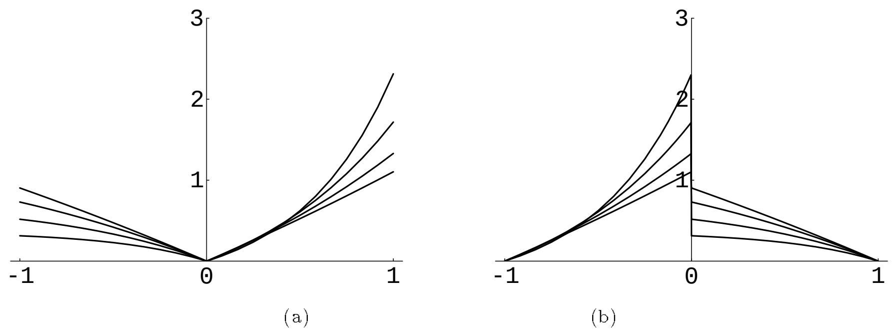


Figure 3: (a) The pdf f_1 of W_T/r for $\varepsilon = 0.1, 0.3, 0.6, 1.0$, with higher ε-values giving higher values of $f_1(1)$ and lower values of $f_1(-1)$. (b) The pdf f_2 of W_U/r for similar ε, with higher ε-values giving higher values of $f_2(0-)$ and lower values of $f_2(0+)$.

(ii) The density f_1 of W_T/r, distributed on [-1,1], is $f_1(x) = |x|$ when $\mu = 0$, otherwise

$$f_1(x) = \frac{\varepsilon}{\mathbf{s^2}[\varepsilon]}\, e^{\varepsilon x}\mathbf{s}[\varepsilon|x|].$$

This can be shown from Theorem 4 by letting $k, m \to \infty$ such that $k/m \to x$, with $|x| \leq 1$, since $m\mathbb{P}\{S_N = k\} \to f_1(x)$. A straightforward calculation verifies that

$$\mathbb{E}(e^{-vW_T/r}) = \int_{-1}^{1} e^{-vx}e^{\varepsilon x}\,\frac{\varepsilon\mathbf{s}[\varepsilon|x|]}{\mathbf{s^2}[\varepsilon]}\,dx,$$

which checks the assertion. The density f_2 of W_U/r is easily derived from that of W_T, namely, for $|x| \le 1$, $f_2(x) = 1 - |x|$ when $\mu = 0$, otherwise

$$f_2(x) = \frac{\varepsilon}{\mathbf{s^2}[\varepsilon]}\, e^{\varepsilon x(1-1/|x|)}\mathbf{s}[\varepsilon(1-|x|)].$$

(iii) Let $\pi(r)$ be the probability that the Brownian motion is at a supremum when the given range r is first attained. Clearly $\pi(r) = \mathbb{P}\{W_T > 0\}$. Either from note (ii) above or from the earlier result for π_m, it is easily shown that, for $\mu \neq 0$,

$$\pi(r) = \frac{(\mathbf{s}[\varepsilon] + \mathbf{c}[\varepsilon])\mathbf{s}[\varepsilon] - \varepsilon}{2\mathbf{s^2}[\varepsilon]},$$

the dependence on r being via $\varepsilon := r\mu/\sigma^2$. Trivially, $\pi(r) = 1/2$ when $\mu = 0$.

(iv) For $u \ge 0$,

$$\mathbb{E}(e^{-uU} \mid W_T = w) = \frac{\mathbf{s}[\varepsilon]}{\mathbf{s}[w\varepsilon/r]}\,\frac{\mathbf{s}[w\alpha(u)/r]}{\mathbf{s}[\alpha(u)]}.$$

By a direct calculation it follows that the expression above satisfies

$$\mathbb{E}(e^{-uU-vW_T}) = \int_{-1}^{1} e^{-vrx}\mathbb{E}(e^{-uU} \mid W_T = rx) f_1(x)\, dx,$$

with $f_1(x)$ as in (ii).

(v) Moments of the random variables T, U, W_T can be derived by differentiations of the Laplace transforms. For example we find for the case with drift,

$$\begin{aligned}
\mathbb{E}(T) &= \frac{1}{\mu}\mathbb{E}(W_T) = \frac{r^2}{2\sigma^2}\Big[\frac{2\coth\varepsilon}{\varepsilon} - \frac{1}{\varepsilon^2} - \frac{1}{\mathbf{s^2}[\varepsilon]}\Big],\\
\mathbb{V}\mathrm{ar}(T) &= \frac{r^4}{4\sigma^4}\Big[-\frac{5}{\varepsilon^4} + \frac{4\coth\varepsilon}{\varepsilon^3} + \frac{6}{\varepsilon^2\mathbf{s^2}\varepsilon]} - \frac{4\coth\varepsilon}{\varepsilon\mathbf{s^2}\varepsilon]} - \frac{1}{\mathbf{s^4}\varepsilon]}\Big].
\end{aligned}$$

For the zero-drift case, $\mathbb{E}(T) = r^2/(2\sigma^2)$ and $\mathbb{V}\mathrm{ar}(T) = r^4/(12\sigma^4)$.

(vi) In the zero-drift case, we note that $\mathbb{E}(e^{-tT}) = 1/\mathbf{c^2}[r\sqrt{t/2}/\sigma]$, a result already known from Imhof [15]. Here $\mathbf{c^n}[x] := \cosh^n(x)$. From series expansion of $1/\mathbf{c^2}[x] = \frac{d}{dx}\tanh x$, we see that the higher moments of T can be expressed in Bernoulli numbers (Abramowitz and Stegun [1, p. 85]).

(vii) The density of Q, defined as the midpoint of the range and distributed on $[-r/2, r/2]$, is

$$f_Q(x) = \frac{\varepsilon e^{2\varepsilon x/r}}{r\mathbf{s}[\varepsilon]}.$$

(viii) Imhof [15] studied the zero-drift case and derived the joint density of T and W_T. After some reorganisation we write his result as, for $t > 0$ and $|w| < r$,

$$f_{T,W_T}(t,w) = \sum_{n=-\infty}^{\infty} n\sqrt{\frac{2}{\pi}}\{t^{-3/2} - (2nr + |w|)^2 t^{-5/2}\}e^{-(2nr+|w|)^2/(2t)}.$$

One can calculate directly from this result the joint Laplace transform of (T, W_T). This is a lengthy calculation. Then one can employ change-of-measure arguments introduced by Girsanov [13] to find the same Laplace transform for the case with drift. We have done this in Chong and Cowan [7], but clearly there is merit in Theorem 7 here, since it involves a third variate and its development is simpler. $\triangle$

5. Distribution of the cover time for Brownian motion

In this section explicit expressions are derived for the distribution of T by inverting its Laplace transform. With $u = t$ and $v = 0$ in Theorem 7 a complicated expression for the Laplace transform is obtained. Using the remarks of that theorem, however, we find a more tractable form.

$$
\begin{aligned}
\mathbb{E}(e^{-tT}) =& \mathbb{E}(e^{-t(T-U)})\,\mathbb{E}(\mathbb{E}(e^{-tU}\,|W_T)) \\
=& \int_{-1}^{1} e^{\varepsilon x}\,\frac{\alpha(t)\mathbf{s}[|x|\alpha(t)]}{\mathbf{s}^2[\alpha(t)]}\,dx \\
=& 2\int_0^1 \mathbf{c}[\varepsilon x]\,\frac{\sqrt{2(\frac{\varepsilon^2}{2}+\frac{r^2}{\sigma^2}t)}\mathbf{s}[x\sqrt{2(\frac{\varepsilon^2}{2}+\frac{r^2}{\sigma^2}t)}]}{\mathbf{s}^2[\sqrt{2(\frac{\varepsilon^2}{2}+\frac{r^2}{\sigma^2}t)}]}\,dx.
\end{aligned}
$$

With $g(x, y)$ such that, for $s > 0$ and $0 < x < 1$,

$$
\frac{\sqrt{2s}\,\mathbf{s}[x\sqrt{2s}]}{\mathbf{s}^2[\sqrt{2s}]} = \int_0^\infty e^{-sy} g(x,y)\,dy,
$$

we obtain

$$
\begin{aligned}
\mathbb{E}(e^{-tT}) =& 2\int_0^1 \mathbf{c}[\varepsilon x]\int_0^\infty e^{-(\frac{\varepsilon^2}{2}+\frac{r^2}{\sigma^2}t)y} g(x,y)\,dy\,dx \\
=& \int_0^\infty e^{-(tr^2/\sigma^2)y}\,2e^{-\varepsilon^2 y/2}\int_0^1 \mathbf{c}[\varepsilon x]\,g(x,y)\,dx\,dy.
\end{aligned}
$$

Thus the density of $\frac{\sigma^2}{r^2}T$ can be written

$$
h(y) = 2e^{-\varepsilon^2 y/2}\int_0^1 \mathbf{c}[\varepsilon x]\,g(x,y)\,dx.
$$

This is calculated below in Theorem 9. Firstly, we determine $g(x, y)$.

Lemma 8. *For $0 < x < 1$ and $y > 0$,*

$$
\begin{aligned}
g(x,y) =& \sum_{n=-\infty}^{\infty} 2n\Big[\big(\frac{2n-x}{\sqrt{y}}\big)^2 - 1\Big]\,y^{-3/2}\varphi\big(\frac{2n-x}{\sqrt{y}}\big) \\
=& \sum_{k=1}^{\infty}\big[y(\pi k)^3\sin(\pi k x) - x(\pi k)^2\cos(\pi k x) - 2\pi k\sin(\pi k x)\big]\,e^{-(\pi k)^2 y/2},
\end{aligned}
$$

where $\varphi(u) := (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}u^2}$ is the standard normal density.

Proof. From Erdélyi [9, Table 5.6.(5), p246] we have, for $\alpha, p > 0$,

$$
\sqrt{p}\,e^{-\sqrt{\alpha p}} = \int_0^\infty e^{-pt}\pi^{-1/2}t^{-5/2}(\alpha/4 - t/2)e^{-\alpha/(4t)}\,dt.
$$

Using this and series expansion we find that

$$\begin{aligned}
\sqrt{2s}\,\frac{\mathbf{s}[x\sqrt{2s}]}{\mathbf{s}^2[\sqrt{2s}]} =& 2\sqrt{2s}\;\frac{e^{-(2-x)\sqrt{2s}}-e^{-(2+x)\sqrt{2s}}}{\left(1-e^{-2\sqrt{2s}}\right)^2}\\
=& 2\sqrt{2s}\sum_{n=1}^{\infty} n\left[e^{-(2n-x)\sqrt{2s}}-e^{-(2n+x)\sqrt{2s}}\right]\\
=& 2\sum_{n=1}^{\infty} n\int_0^\infty e^{-sy}\Big[\frac{(2n-x)^2-y}{y^{5/2}}\varphi\big(\frac{2n-x}{\sqrt{y}}\big)\\
&\qquad -\frac{(2n+x)^2-y}{y^{5/2}}\varphi\big(\frac{2n+x}{\sqrt{y}}\big)\Big]\,dy\\
=& \int_0^\infty e^{-sy}\sum_{n=-\infty}^{\infty} 2n\Big[\big(\frac{2n-x}{\sqrt{y}}\big)^2-1\Big]\,y^{-3/2}\varphi\big(\frac{2n-x}{\sqrt{y}}\big)\,dy\,,
\end{aligned}$$

thereby proving the first equality in the assertion. For the second equality we write

$$g(x,y)=\sum_{n=-\infty}^{\infty}\Big[\frac{1}{y}\Big\{\big(\frac{2n-x}{\sqrt{y}}\big)^3-\frac{2n-x}{\sqrt{y}}\Big\}+\frac{x}{y^{3/2}}\Big\{\big(\frac{2n-x}{\sqrt{y}}\big)^2-1\Big\}\Big]\,\varphi\big(\frac{2n-x}{\sqrt{y}}\big).$$

Next we use the Poisson summation formula (see Dym and McKean [8, p111]). By the moment generating function of the standard normal distribution we have

$$\int_{-\infty}^{\infty} e^{tv}v^j\,\varphi(v)dv=\frac{d^j}{dt^j}e^{t^2/2}.$$

Using this one can readily obtain that the Fourier transforms

$$c_j(k)=\int_{-\infty}^{\infty} e^{-2\pi ikz}\,\big(\frac{2z-x}{\sqrt{y}}\big)^j\,\varphi\big(\frac{2z-x}{\sqrt{y}}\big)\,dz,\quad j=0,1,2,3,$$

are given by $c_0(k)=\frac{1}{2}\sqrt{y}\,e^{-i\pi kx-(\pi k)^2y/2}$, $c_1(k)=-i\pi k\sqrt{y}\,c_0(k)$, $c_2(k)=(1-(\pi k)^2y)\,c_0(k)$ and $c_3(k)=(-i3\pi k\sqrt{y}+i(\pi k)^3y^{3/2})\,c_0(k)$. The Poisson summation formula gives, after some simple manipulations,

$$\begin{aligned}
g(x,y)=&\sum_{k=-\infty}^{\infty}\Big[\frac{1}{y}\big(c_3(k)-c_1(k)\big)+\frac{x}{y^{3/2}}\big(c_2(k)-c_0(k)\big)\Big]\\
=&\sum_{k=1}^{\infty}\big[y(\pi k)^3\sin(\pi kx)-x(\pi k)^2\cos(\pi kx)-2\pi k\sin(\pi kx)\big]e^{-(\pi k)^2y/2}\,,
\end{aligned}$$

thus proving the second equality in the assertion. $\square$

Theorem 9. *Let T be the time until a Brownian motion with drift μ and variance $\sigma^2 > 0$ attains a given range r. Set $\varepsilon := \frac{r\mu}{\sigma^2}$. The density of $\frac{\sigma^2}{r^2}T$ can be written as*

$$\begin{aligned} h(y) =& 4e^{-\varepsilon^2 y/2}\sum_{n=1}^{\infty}\Big[\mathbf{c}[\varepsilon]\,\frac{(2n-1)^2}{y^{3/2}}\varphi\big(\frac{2n-1}{\sqrt{y}}\big) - \frac{(2n)^2}{y^{3/2}}\varphi\big(\frac{2n}{\sqrt{y}}\big) - \frac{\varepsilon\mathbf{s}[\varepsilon]}{\sqrt{y}}\varphi\big(\frac{2n-1}{\sqrt{y}}\big)\Big] \\ &+ \varepsilon^2\sum_{n=-\infty}^{\infty} 2n\,e^{2n\varepsilon}\Big[2\Phi\big(\frac{2n+\varepsilon y}{\sqrt{y}}\big) - \Phi\big(\frac{2n+1+\varepsilon y}{\sqrt{y}}\big) - \Phi\big(\frac{2n-1+\varepsilon y}{\sqrt{y}}\big)\Big] \\ =& 2e^{-\varepsilon^2 y/2}\sum_{k=1}^{\infty}\frac{(\pi k)^2}{\varepsilon^2+(\pi k)^2}\Big[(\pi k)^2\big(1-(-1)^k\mathbf{c}[\varepsilon]\big)y \\ &- \frac{3\varepsilon^2+(\pi k)^2}{\varepsilon^2+(\pi k)^2}\big(1-(-1)^k\mathbf{c}[\varepsilon]\big) - (-1)^k\,\varepsilon\mathbf{s}[\varepsilon]\Big]\,e^{-(\pi k)^2 y/2}, \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Proof. We obtain using integration by parts

$$\begin{aligned} &\int_0^1 \mathbf{c}[\varepsilon x]\,\Big[\big(\frac{x-2n}{\sqrt{y}}\big)^2 - 1\Big]y^{-3/2}\varphi\big(\frac{x-2n}{\sqrt{y}}\big)\,dx \\ =&\mathbf{c}[\varepsilon]\,\frac{2n-1}{y^{3/2}}\varphi\big(\frac{2n-1}{\sqrt{y}}\big) - \frac{2n}{y^{3/2}}\varphi\big(\frac{2n}{\sqrt{y}}\big) + \frac{\varepsilon}{2}\int_0^1\big(e^{\varepsilon x} - e^{-\varepsilon x}\big)\frac{x-2n}{y^{3/2}}\varphi\big(\frac{x-2n}{\sqrt{y}}\big)\,dx. \end{aligned}$$

Furthermore,

$$\int_0^1 e^{\varepsilon x}\frac{x-2n}{y^{3/2}}\varphi\big(\frac{x-2n}{\sqrt{y}}\big)e^{-\varepsilon^2 y/2}\,dx = e^{2n\varepsilon}\int_{-2n}^{-2n+1}\big(\frac{x-\varepsilon y}{y^{3/2}} + \frac{\varepsilon}{\sqrt{y}}\big)\varphi\big(\frac{x-\varepsilon y}{\sqrt{y}}\big)\,dx$$

$$= e^{2n\varepsilon}\Big[\frac{1}{\sqrt{y}}\Big\{\varphi\big(\frac{2n+\varepsilon y}{\sqrt{y}}\big) - \varphi\big(\frac{2n-1+\varepsilon y}{\sqrt{y}}\big)\Big\} + \varepsilon\Big\{\Phi\big(\frac{2n+\varepsilon y}{\sqrt{y}}\big) - \Phi\big(\frac{2n-1+\varepsilon y}{\sqrt{y}}\big)\Big\}\Big].$$

Hence, by Lemma 8 and using symmetry we get

$$\begin{aligned} h(y) =& \sum_{n=-\infty}^{\infty} 4n\,e^{-\varepsilon^2 y/2}\int_0^1 \mathbf{c}[\varepsilon x]\Big[\big(\frac{x-2n}{\sqrt{y}}\big)^2 - 1\Big]y^{-3/2}\varphi\big(\frac{x-2n}{\sqrt{y}}\big)\,dx \\ =& 2\sum_{n=-\infty}^{\infty}\Big[\mathbf{c}[\varepsilon]\,\frac{(2n-1)^2}{y^{3/2}}\varphi\big(\frac{2n-1}{\sqrt{y}}\big) - \frac{(2n)^2}{y^{3/2}}\varphi\big(\frac{2n}{\sqrt{y}}\big)\Big]e^{-\varepsilon^2 y/2} \\ &+ \varepsilon\sum_{n=-\infty}^{\infty} 2n\,e^{2n\varepsilon}\Big[\frac{1}{\sqrt{y}}\Big\{2\varphi\big(\frac{2n+\varepsilon y}{\sqrt{y}}\big) - \varphi\big(\frac{2n+1+\varepsilon y}{\sqrt{y}}\big) - \varphi\big(\frac{2n-1+\varepsilon y}{\sqrt{y}}\big)\Big\} \\ &+ \varepsilon\Big\{2\Phi\big(\frac{2n+\varepsilon y}{\sqrt{y}}\big) - \Phi\big(\frac{2n+1+\varepsilon y}{\sqrt{y}}\big) - \Phi\big(\frac{2n-1+\varepsilon y}{\sqrt{y}}\big)\Big\}\Big]. \end{aligned}$$

Now

$$\sum_{n=-\infty}^{\infty} 2n\, e^{2n\varepsilon}\Big[2\varphi(\frac{2n+\varepsilon y}{\sqrt{y}}) - \varphi(\frac{2n+1+\varepsilon y}{\sqrt{y}}) - \varphi(\frac{2n-1+\varepsilon y}{\sqrt{y}})\Big]$$

$$= \sum_{n=-\infty}^{\infty} 2n\Big[2\varphi(\frac{2n}{\sqrt{y}}) - \varphi(\frac{2n+1}{\sqrt{y}})e^{-\varepsilon} - \varphi(\frac{2n-1}{\sqrt{y}})e^{\varepsilon}\Big]e^{-\varepsilon^2 y/2}$$

$$= \sum_{n=-\infty}^{\infty} \Big[\varphi(\frac{2n+1}{\sqrt{y}})e^{-\varepsilon} - \varphi(\frac{2n-1}{\sqrt{y}})e^{\varepsilon}\Big]e^{-\varepsilon^2 y/2} = -4\mathbf{s}[\varepsilon]\sum_{n=1}^{\infty}\varphi(\frac{2n-1}{\sqrt{y}})e^{-\varepsilon^2 y/2},$$

and the first equality in the assertion follows. For the second representation of the density we use Lemma 8 to get

$$h(y) = 2e^{-\varepsilon^2 y/2}\sum_{k=1}^{\infty} e^{-(\pi k)^2 y/2}\int_0^1 \mathbf{c}[\varepsilon x]\big[y(\pi k)^3\sin(\pi k x) - x(\pi k)^2\cos(\pi k x) - 2\pi k\sin(\pi k x)\big]\,dx\,.$$

Some elementary integrations gives the second equality in the assertion. □

Remarks. (i) By integrating the second representation of the density, the distribution function of the cover time T can be obtained.

(ii) The distributions of U and $T-U$ can be derived in much the same way as that of T.

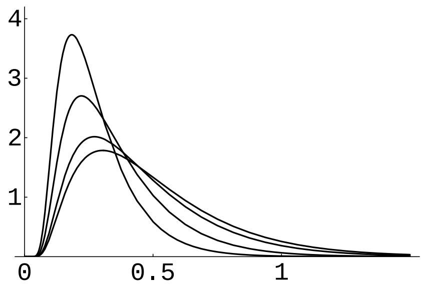
(iii) From Theorem 9 we obtain the distribution function of the *range* of a Brownian motion at a *fixed* time t, $R_t := \sup_{0\le s\le t} W_s - \inf_{0\le s\le t} W_s$, as

$$\mathbb{P}\{R_t \le r\} = \mathbb{P}\{R_t < r\} = \mathbb{P}\{T_r > t\} = \int_{t\sigma^2/r^2}^{\infty} h_r(y)\,dy.$$

To date we have not suffixed T or h by r, partly because in our expressions for h the variable r is not visible, being subsumed within ε. Now, emphasising that h as seen in Theorem 9 depends on r, we find the density of R_t, for $r > 0$, to be

$$\begin{aligned}
g_t(r) &= \frac{2t\sigma^2}{r^3}\,h_r\Big(\frac{t\sigma^2}{r^2}\Big) + \int_{t\sigma^2/r^2}^{\infty}\frac{\partial}{\partial r}h_r(y)\,dy \\
&= \frac{4t\sigma^6}{r^3}\,e^{-t\mu^2/(2\sigma^2)}\sum_{k=1}^{\infty}\frac{(\pi k)^2}{(r\mu)^2+(\pi k\sigma^2)^2}\Big[\big(1-(-1)^k\mathbf{c}[\tfrac{r\mu}{\sigma^2}]\big)\frac{(\pi k\sigma)^2 t}{r^2} \\
&\qquad - (-1)^k\frac{r\mu}{\sigma^2}\mathbf{s}[\tfrac{r\mu}{\sigma^2}] - \frac{3(r\mu)^2+(\pi k\sigma^2)^2}{(r\mu)^2+(\pi k\sigma^2)^2}\big(1-(-1)^k\mathbf{c}[\tfrac{r\mu}{\sigma^2}]\big)\Big]\,e^{-(\pi k\sigma)^2 t/(2r^2)} \\
&\quad + 4\mu\sigma^4 e^{-t\mu^2/(2\sigma^2)}\sum_{k=1}^{\infty}\frac{(\pi k)^2}{[(r\mu)^2+(\pi k\sigma^2)^2]^2}\Big[r\mu\Big\{\frac{12\sigma^4[(r\mu)^2-(\pi k\sigma^2)^2]}{[(r\mu)^2+(\pi k\sigma^2)^2]^2} \\
&\quad - (-1)^k r\mu\mathbf{c}[\tfrac{r\mu}{\sigma^2}] + \frac{\sigma^2 t[3(r\mu)^2-5(\pi k\sigma^2)^2]}{r^2[(r\mu)^2+(\pi k\sigma^2)^2]} - \frac{(\pi k\sigma^2 t)^2}{r^4}\Big\}\big(1-(-1)^k\mathbf{c}[\tfrac{r\mu}{\sigma^2}]\big) \\
&\quad + (-1)^k\mathbf{s}[\tfrac{r\mu}{\sigma^2}]\Big\{\frac{t[(r\mu)^2-(\pi k\sigma^2)^2]}{r^2} + \frac{2\sigma^2[3(r\mu)^2-(\pi k\sigma^2)^2]}{(r\mu)^2+(\pi k\sigma^2)^2}\Big\}\Big]\,e^{-(\pi k\sigma)^2 t/(2r^2)}.
\end{aligned}$$

For the zero-drift case, $\mu = 0$, the density of R_t is derived in Feller [11] and also in Imhof [15]; it is also listed in the Borodin and Salminen handbook [5, p.124 & p.147]. Our distributional result for the case with drift seems, however, to be new. Figure 4(b) shows the effect of drift on the density of the range, whilst Figure 4(a) shows the same phenomenon for the cover time.



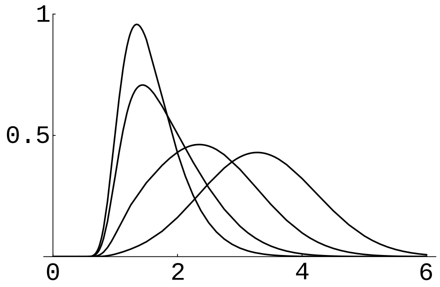


Figure 4: (a) The pdf of the cover time T when $r = 1$ and $\sigma = 1$ for $\mu = 0, 1, 2, 3$; note that the distribution shifts to the left as the drift μ increases. (b) The pdf of the range R_t when $t = 1$ and $\sigma = 1$ for similar μ, with the distribution now shifting to the right as the drift μ increases.

(iv) By integrating in Theorem 9 we obtain a new form for the distribution of T in the zero-drift case, standardised with unit variance for direct comparison with Imhof [15]. In that case

$$\begin{aligned}\mathbb{P}\{T > r^2 y\} &= 8 \sum_{n=1}^{\infty} \Big(y + \frac{1}{(2n-1)^2\pi^2} \Big)\, e^{-(2n-1)^2\pi^2 y/2},\\ \mathbb{P}\{T < r^2 y\} &= 8 \sum_{n=1}^{\infty} (-1)^{n+1} n \Big(1 - \Phi\big(\frac{n}{\sqrt{y}}\big) \Big),\end{aligned}$$

and the density of $\frac{T}{r^2}$ is

$$h(y) = 4 \sum_{n=1}^{\infty} \Big((2n-1)^2\pi^2 y - 1 \Big)\, e^{-(2n-1)^2\pi^2 y/2} = 4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{y^{3/2}}\, \varphi\big(\frac{n}{\sqrt{y}}\big). \tag{4}$$

The series in the formula for $\mathbb{P}\{T > r^2 y\}$ converges very rapidly unless y is small, while the series for $\mathbb{P}\{T < r^2 y\}$ is useful for small y.

For general r in the zero-drift, unit-variance case, Imhof [15] gives the density f_T in the following form that can be compared to our h in (4):

$$f_T(y) = \frac{2}{\sqrt{2\pi y}} \frac{\partial}{\partial r} \sum_{n=-\infty}^{\infty} (-1)^n \exp\big(\frac{-n^2 r^2}{2y}\big).$$

△

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