SHAPES OF RECTANGULAR PRISMS AFTER REPEATED RANDOM DIVISION

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Abstract

The shape of a rectangular prism in $(d+1)$-dimensions is defined as $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_d)$, $Y_n = L_n/L_{n+1}$ where the L_n are the prism's edge lengths, in ascending order. We investigate shape distributions which are invariant when the prism is cut into two, also rectangular, prisms, with one prism retained for measurement and the other discarded. Interesting new distributions on $[0,1]^d$ arise.

KEYWORDS: Invariant distributions, shape distributions, random geometry, distribution theory, Markov processes.

1. Introduction

Some interest has been shown in Markovian sequences of geometric shapes. Mostly the focus has been on triangular shapes in the plane, rather natural random-geometric constructions creating the next triangle in the sequence from its predecessor (Watson 1986; Mannion 1990, 1993; Miles 1983). Sequences of other geometric figures have not been explored in much detail, not through lack of interest but due to mathematical difficulty. This paper deals with shape sequences of rectangular prisms; the analysis and results are presented as a contribution to the theory of Markovian shape sequences, itself a part of the larger probabilistic theory of shape initiated by Kendall (1977, 1984) and, in a rather different way, by Bookstein (1978).

Consider a rectangular prism in $(d+1)$-dimensional Euclidean space, where $d \geq 1$, with one vertex labelled O. This vertex has $d+1$ orthogonal edges emanating from it. The lengths of these edges in ascending order are $L_1, L_2, \cdots, L_{d+1}$, with L_{d+1} being arbitrarily set equal to 1. The shape of this prism can be defined by d quantities involving ratios of consecutive edge-lengths from this ordered list, namely by the d-dimensional vector $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_d)$, where $Y_n = L_n/L_{n+1}$. Thus each component of shape is a number $\in [0,1]$.

The prism is to be split by a randomly-positioned hyperplane of dimension d orthogonal to a randomly-chosen edge, edge j is selected with probability p_j, $j =$

$1, 2, \cdots, d+1$. A point T is then placed on the chosen edge j such that U, defined as $|OT|/L_j$, is randomly sampled according to a distribution function G on $[0,1]$. A d-dimensional (hyper)planar cut orthogonal to edge j is then made. The prism containing O is retained, whilst the other prism is discarded.

The retained prism is rescaled so that its longest edge equals 1 and the process then iterated, thus forming a Markov process of shape vectors. In the first part of this paper, we investigate the equilibrium conditions and equilibrium distribution for this Markov process. We have found it convenient to use, for this investigation, $G(u) = u^a$ for some parameter $a > 0$, because results for this family are elegant and simply-expressed; this family includes the case of a uniform distribution for the split position. Our main results are that (i) a non-degenerate equilibrium shape distribution on $[0,1]^d$ exists if $\sum_{i=1}^{n} p_i < n/(d+1)$ for $n = 1, 2, \cdots, d$, (ii) the d components of shape at this equilibrium are independent random variables and (iii) the equilibrium distribution of Y_n, the nth component, has probability distribution function described by $P\{Y_n \leq y_n\} = y_n^{\beta_n}$, where $\beta_n = a[n - (d+1)\sum_{i=1}^{n} p_i]$.

In the second part of the paper, we modify the splitting rule. At each stage of the process, we consider the set $\mathcal{H}$ of all d-dimensional hyperplanes which can cut the prism into two rectangular 'daughter' prisms. We then sample a hyperplane in an equally-likely manner from $\mathcal{H}$. This means that the probability p_j, which now equals $L_j/(L_1 + L_2 + \cdots + L_{d+1})$, depends on the shape of the prism to be cut whilst the position of the cut on the selected edge is uniformly distributed on this edge.

Our results show that an equilibrium distribution on $[0,1]$ exists, with its joint probability density function taking the form

$$\frac{(d+1)!d!y_2 y_3^2 y_4^3 \cdots y_d^{d-1}}{(1 + y_d + y_d y_{d-1} + y_d y_{d-1} y_{d-2} + \cdots + y_d y_{d-1} \cdots y_2 y_1)^{d+1}}. \tag{1}$$

This generalises a result for the case $d = 1$ mentioned by Miles (1983).

2. Analysis of the rule with G and $p_1, p_2, \cdots, p_{d+1}$

First, we establish that the distribution mentioned in (iii) above is invariant under the first-mentioned rules for dividing the prism. Suppose a prism with shape $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_d)$ and longest edge of length 1 is divided according to the random rules. Let j be the index of the edge which is cut and let $\boldsymbol{X} = (X_1, X_2, \cdots, X_d)$ be the shape of the prism retained after the cut.

The greatest economy in writing the random transitions from $\boldsymbol{Y}$ to $\boldsymbol{X}$ comes from a categorisation based on j. For $j = 1$, $X_1 = UY_1$ for all $d \geq 1$, whilst $X_n = Y_n$ for $2 \leq n \leq d$. For $j = 2$, $X_1 = \min[Y_1/U, U/Y_1]$ for all $d \geq 1$, $X_2 = Y_2 \max[U, Y_1]$ for

$d \geq 2$, whilst $X_n = Y_n$ for $3 \leq n \leq d$. When $j = 3$,

$$
\begin{aligned}
(X_1, X_2) &= \big(Y_1, \frac{Y_2}{U}\big) && d \geq 2,\ U > Y_2, \\
&= \big(\frac{Y_1 Y_2}{U}, \frac{U}{Y_2}\big) && d \geq 2,\ Y_1 Y_2 < U \leq Y_2, \\
&= \big(\frac{U}{Y_1 Y_2}, Y_1\big) && d \geq 2,\ U \leq Y_1 Y_2. \\
X_3 &= Y_3 \max[U, Y_2] && d \geq 3. \\
X_n &= Y_n && 4 \leq n \leq d.
\end{aligned}
$$

The cases $d = 1$ and $d = 2$ can be extracted from the information above. For example, when $d = 1$, X_1 equals either UY_1 or $\min[Y_1/U, U/Y_1]$ as $j = 1$ or $j = 2$ respectively. When $j = 4$ and $d \geq 3$,

$$
\begin{aligned}
(X_1, X_2, X_3) &= \big(Y_1, Y_2, \frac{Y_3}{U}\big) && d \geq 3,\ U > Y_3, \\
&= \big(Y_1, \frac{Y_2 Y_3}{U}, \frac{U}{Y_3}\big) && d \geq 3,\ Y_2 Y_3 < U \leq Y_3, \\
&= \big(\frac{Y_1 Y_2 Y_3}{U}, \frac{U}{Y_2 Y_3}, Y_2\big) && d \geq 3,\ Y_1 Y_2 Y_3 < U \leq Y_2 Y_3, \\
&= \big(\frac{U}{Y_1 Y_2 Y_3}, Y_1, Y_2\big) && d \geq 3,\ U \leq Y_1 Y_2 Y_3. \\
X_4 &= Y_4 \max[U, Y_3] && d \geq 4. \\
X_n &= Y_n && 5 \leq n \leq d.
\end{aligned}
$$

General formulae emerge for $j \geq 5$, $d \geq j - 1$,

$$
\begin{aligned}
(\boldsymbol{X}_1^{j-1}) &= \big(\boldsymbol{Y}_1^{j-2}, \frac{Y_{j-1}}{U}\big) && U > Y_{j-1}, \\
&= \big(\boldsymbol{Y}_1^{j-3}, \frac{Y_{j-2} Y_{j-1}}{U}, \frac{U}{Y_{j-1}}\big) && Y_{j-2} Y_{j-1} < U < Y_{j-1}, \\
&= \cdots \\
&= \big(\boldsymbol{Y}_1^{j-k}, \frac{Y_{j-k+1} \cdots Y_{j-1}}{U}, \frac{U}{Y_{j-k+2} \cdots Y_{j-1}}, \boldsymbol{Y}_{j-k+2}^{j-2}\big) \\
&= \cdots \\
&= \big(\frac{Y_1 \cdots Y_{j-1}}{U}, \frac{U}{Y_2 \cdots Y_{j-1}}, \boldsymbol{Y}_2^{j-2}\big) && Y_1 \cdots Y_{j-1} < U < Y_2 \cdots Y_{j-1} \\
&= \big(\frac{U}{Y_1 \cdots Y_{j-1}}, \boldsymbol{Y}_1^{j-2}\big) && U < Y_1 \cdots Y_{j-1},
\end{aligned}
$$

where $\boldsymbol{Y}_n^m$, $n \le m$, is shorthand for $Y_n, Y_{n+1}, \cdots, Y_m$; $\boldsymbol{X}_n^m$ is defined similarly. The central expression above has range $Y_{j-k+1} \cdots Y_{j-1} < U \le Y_{j-k+2} \cdots Y_{j-1}$, with a version for each k in the range $4 \le k \le j-1$. In addition, for $j \ge 5$, as for lower values of j, $X_j = Y_j \max[U, Y_{j-1}]$ when $d \ge j$, whilst $X_n = Y_n$ for $j+1 \le n \le d$.

Suppose that the side j selected, the shape vector $\boldsymbol{Y}$ and the cut positioner U are random variables in $\mathcal{T} := \{1, 2, \cdots, (d+1)\} \times [0,1]^{d+1}$ with known distribution. The formulae above, combined with the mapping $S = U$, give different transformations to $(\boldsymbol{X}, S)$ in different parts of $\mathcal{T}$. Note from the formulae above that, for each j, the range of $(\boldsymbol{Y}, U)$ is broken up into j disjoint parts, $\{U > Y_{j-1}\}, \{Y_{j-2}Y_{j-1} < U \le Y_{j-1}\}, \cdots$. Denote these parts by $A_{j,m}$, $m = 1, 2, \cdots, j$, and the transformation operative in $A_{j,m}$ by $t_{j,m} : A_{j,m} \to B_{j,m} \subset [0,1]^{d+1}$. For example, $A_{3,2} = \{Y_1 Y_2 < U \le Y_2\}$ and, if $d = 4$ say, then $t_{3,2}$ maps (Y_1, Y_2, Y_3, Y_4, U) to $(X_1, X_2, X_3, X_4, S) = (Y_1 Y_2 / U, U/Y_2, Y_2 Y_3, Y_4, U)$. These transformations have inverses from $B_{j,m}$ to $A_{j,m}$. For example, $B_{3,2} = \{X_2 X_3 < S \le X_2\} \subset [0,1]^5$, whilst $t_{3,2}^{-1}$ maps (X_1, X_2, X_3, X_4, S) to $(Y_1, Y_2, Y_3, Y_4, U) = (X_1 X_2, S/X_2, X_2 X_3 / S, X_4, S)$. The Jacobian of this map is 1 (though in other cases, different values occur).

If h is the probability density function (pdf) of $(\boldsymbol{Y}, U)$ and $q_{j,m}$ is the probability, given j, of the event $\{(\boldsymbol{Y}, U) \in A_{j,m}\}$, then the pdf $a_{j,m}$ of $(\boldsymbol{X}, S)$ conditional upon this event is $J_{j,m} h(t_{j,m}^{-1}(\boldsymbol{x}, s))/q_{j,m}$ on $B_{j,m}$, but zero elsewhere. Here $J_{j,m}$ is the Jacobian of the relevant transformation. Overall, the pdf of $(\boldsymbol{X}, S)$ is $\sum_{j=1}^{d+1} \sum_{m=1}^{j} p_j q_{j,m} a_{j,m}(\boldsymbol{x}, s)$, or $\sum_{j=1}^{d+1} \sum_{m=1}^{j} p_j J_{j,m} h(t_{j,m}^{-1}(\boldsymbol{x}, s)) \chi\{(\boldsymbol{x}, s) \in B_{j,m}\}$, where we introduce the indicator function $\chi\{\cdot\}$ to show that the individual terms contribute zero outside their particular $B_{j,m}$. Integration over all s gives the joint pdf, f say, of $\boldsymbol{X}$. This procedure yields, after recognising that $h(\boldsymbol{y}, u) = f(\boldsymbol{y}) g(u)$ where $g = G'$, an equation in f for each value of d. For the 2-dimensional case $d = 1$, this equation is (with x for x_1)

$$f(x) = p_1 \int_x^1 \frac{f(x/s)g(s)}{s} ds + p_2 \left[\frac{1}{x^2} \int_0^x s f(s/x) g(s) ds + \int_0^1 s f(sx) g(s) ds \right]. \tag{2}$$

For the 3-dimensional case $d = 2$, the equation is (using x for x_1, y for x_2)

$$\begin{aligned} f(x,y) = {} & p_1 \int_x^1 \frac{f(x/s, y)g(s)}{s} ds \\ & + p_2 \left[\int_y^1 f(sx, y/s) g(s) ds + \frac{1}{x} \int_{xy}^x f(s/x, xy/s) g(s) ds \right] \\ & + p_3 \left[\frac{1}{x^2 y} \int_0^{xy} s f(y, s/xy) g(s) ds + \frac{1}{y} \int_0^y s f(xy, s/y) g(s) ds \right. \end{aligned}$$

$$+\int_0^1 sf(x,sy)g(s)ds\Bigg]. \tag{3}$$

The equations become progressively more complicated as d increases. We show the equation for $d>5$, where a general form emerges. The cases for $3\le d\le 5$ are not displayed here. Using, when necessary for typesetting reasons, the shorthand notations $\boldsymbol{x}_n^m := x_n, x_{n+1}, \cdots, x_m$ and $x_{n,m} := x_n x_{n+1}\cdots x_m$ for $n\le m$, we have for $d>5$

$$
\begin{aligned}
f(\boldsymbol{x}) = {} & p_1 \int_{x_1}^1 f\big(\frac{x_1}{s}, \boldsymbol{x}_2^d\big) g(s)\frac{ds}{s} \\
& + p_2\left[\int_{x_2}^1 f\big(sx_1, \frac{x_2}{s}, \boldsymbol{x}_3^d\big)g(s)ds + \frac{1}{x_1}\int_{x_1x_2}^{x_1} f\big(\frac{s}{x_1}, \frac{x_1x_2}{s}, \boldsymbol{x}_3^d\big)g(s)ds\right] \\
& + p_3\left[\int_{x_3}^1 f\big(x_1, sx_2, \frac{x_3}{s}, \boldsymbol{x}_4^d\big)g(s)ds + \int_{x_2x_3}^{x_2} f\big(x_1x_2, \frac{s}{x_2}, \frac{x_2x_3}{s}, \boldsymbol{x}_4^d\big)g(s)ds\right. \\
& \left. + \frac{1}{x_1}\int_{x_1x_2x_3}^{x_1x_2} f\big(x_2, \frac{s}{x_1x_2}, \frac{x_1x_2x_3}{s}, \boldsymbol{x}_4^d\big)g(s)ds\right] \\
& + p_4\left[\int_{x_4}^1 f\big(x_1, x_2, sx_3, \frac{x_4}{s}, \boldsymbol{x}_5^d\big)g(s)ds + \int_{x_3x_4}^{x_3} f\big(x_1, x_2x_3, \frac{s}{x_3}, \frac{x_3x_4}{s}, \boldsymbol{x}_5^d\big)g(s)ds\right. \\
& + \int_{x_2x_3x_4}^{x_2x_3} f\big(x_1x_2, x_3, \frac{s}{x_2x_3}, \frac{x_2x_3x_4}{s}, \boldsymbol{x}_5^d\big)g(s)ds \\
& \left. + \frac{1}{x_1}\int_{x_{1,4}}^{x_{1,3}} f\big(x_2, x_3, \frac{s}{x_{1,3}}, \frac{x_{1,4}}{s}, \boldsymbol{x}_5^d\big)g(s)ds\right] \\
& + \sum_{j=5}^{d-1} p_j \left[\int_{x_j}^1 f\big(\boldsymbol{x}_1^{j-2}, sx_{j-1}, \frac{x_j}{s}, \boldsymbol{x}_{j+1}^d\big)g(s)ds\right. \\
& + \int_{x_{j-1}x_j}^{x_{j-1}} f\big(\boldsymbol{x}_1^{j-3}, x_{j-2}x_{j-1}, \frac{s}{x_{j-1}}, \frac{x_{j-1}x_j}{s}, \boldsymbol{x}_{j+1}^d\big)g(s)ds \\
& + \sum_{k=4}^{j-1}\left\{\int_{x_{j-k+2,j}}^{x_{j-k+2,j-1}} f\big(\boldsymbol{x}_1^{j-k}, x_{j-k+1}x_{j-k+2}, \boldsymbol{x}_{j-k+3}^{j-1}, \frac{s}{x_{j-k+2,j-1}}, \frac{x_{j-k+2,j}}{s}, \boldsymbol{x}_{j+1}^d\big)g(s)ds\right\} \\
& + \int_{x_{2,j}}^{x_{2,j-1}} f\big(x_1x_2, \boldsymbol{x}_3^{j-1}, \frac{s}{x_{2,j-1}}, \frac{x_{2,j}}{s}, \boldsymbol{x}_{j+1}^d\big)g(s)ds \\
& \left. + \frac{1}{x_1}\int_{x_{1,j}}^{x_{1,j-1}} f\big(\boldsymbol{x}_2^{j-1}, \frac{s}{x_{1,j-1}}, \frac{x_{1,j}}{s}, \boldsymbol{x}_{j+1}^d\big)g(s)ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ p_d \Bigg[\int_{x_d}^{1} f\big(\boldsymbol{x}_1^{d-2}, s x_{d-1}, \frac{x_d}{s}\big) g(s) ds + \int_{x_{d-1}x_d}^{x_{d-1}} f\big(\boldsymbol{x}_1^{d-3}, x_{d-2}x_{d-1}, \frac{s}{x_{d-1}}, \frac{x_{d-1}x_d}{s}\big) g(s) ds \\
&+ \sum_{k=4}^{d-1} \Bigg\{ \int_{x_{d-k+2,d}}^{x_{d-k+2,d-1}} f\big(\boldsymbol{x}_1^{d-k}, x_{d-k+1}x_{d-k+2}, \boldsymbol{x}_{d-k+3}^{d-1}, \frac{s}{x_{d-k+2,d-1}}, \frac{x_{d-k+2,d}}{s}\big) g(s) ds \Bigg\} \\
&+ \int_{x_{2,d}}^{x_{2,d-1}} f\big(x_1 x_2, \boldsymbol{x}_3^{d-1}, \frac{s}{x_{2,d-1}}, \frac{x_{2,d}}{s}\big) g(s) ds \\
&+ \frac{1}{x_1} \int_{x_{1,d}}^{x_{1,d-1}} f\big(\boldsymbol{x}_2^{d-1}, \frac{s}{x_{1,d-1}}, \frac{x_{1,d}}{s}\big) g(s) ds \Bigg] \\
&+ p_{d+1} \Bigg[\frac{1}{x_1^2 x_{2,d}} \int_0^{x_{1,d}} s f\big(\boldsymbol{x}_2^d, \frac{s}{x_{1,d}}\big) g(s) ds + \frac{1}{x_{2,d}} \int_0^{x_{2,d}} s f\big(x_1 x_2, \boldsymbol{x}_3^d, \frac{s}{x_{2,d}}\big) g(s) ds \\
&+ \sum_{k=3}^{d-1} \Bigg\{ \frac{1}{x_{k,d}} \int_0^{x_{k,d}} s f\big(\boldsymbol{x}_1^{k-2}, x_{k-1}x_k, \boldsymbol{x}_{k+1}^d, \frac{s}{x_{k,d}}\big) g(s) ds \Bigg\} \\
&+ \frac{1}{x_d} \int_0^{x_d} s f\big(\boldsymbol{x}_1^{d-2}, x_{d-1}x_d, \frac{s}{x_d}\big) g(s) ds + \int_0^1 s f\big(\boldsymbol{x}_1^{d-1}, s x_d\big) g(s) ds \Bigg]. \qquad (4)
\end{aligned}
$$

Substitution of $g(s) = as^{a-1}$ and $f(y_1, y_2, \cdots, y_d) = \prod_{i=1}^{d} \beta_i y_i^{\beta_i - 1}$ produces simple expressions for each of the integrals in (4), provided

$$
\begin{aligned}
\theta_{d+1} &:= a + \beta_d \neq 0 \\
\theta_n &:= a + \beta_{n-1} - \beta_n \neq 0 \qquad 2 \leq n \leq d \\
\theta_1 &:= a - \beta_1 \neq 0,
\end{aligned} \qquad (5)
$$

the condition for θ_n not being needed if the corresponding $p_n = 0$. Assuming (5) for the moment, the RHS of (4) becomes $a \prod_{i=1}^{d} \beta_i x_i^{\beta_i - 1}$ times

$$
\begin{aligned}
&\frac{p_1(1 - x_1^{\theta_1})}{\theta_1} + \sum_{n=2}^{d} \frac{p_n(1 - x_n^{\theta_n})}{\theta_n} \Big\{ 1 + x_{n-1}^{\theta_{n-1}} \{ 1 + x_{n-2}^{\theta_{n-2}} \{ 1 + \cdots \{ 1 + x_1^{\theta_1} \} \cdots \} \} \Big\} \\
&+ \frac{p_{d+1}}{\theta_{d+1}} \Big\{ 1 + x_d^{\theta_d} \{ 1 + d_{d-1}^{\theta_{d-1}} \{ 1 + \cdots \{ 1 + x_1^{\theta_1} \} \cdots \} \} \Big\}.
\end{aligned} \qquad (6)
$$

Consider firstly the case where all $p_n > 0$ and where (5) is assumed for all n. If our chosen f is to be a solution of (4), then (6) must equal $1/a$. By equating the constant term of (6) to $1/a$ and coefficients of other terms to zero, one has

$$\sum_{n=1}^{d+1} \frac{p_n}{\theta_n} = \frac{1}{a} \tag{7}$$

$$\frac{p_1}{\theta_1} = \frac{p_2}{\theta_2} = \cdots = \frac{p_{d+1}}{\theta_{d+1}}. \tag{8}$$

It is thus evident that f is a solution of (4) if, for all n,

$$\theta_n = a(d+1)p_n. \tag{9}$$

When some of the p_n ($n \leq d$) are zero, the exercise which previously led to (7) and (8) requires modification. One can show that, when $p_n = 0$ and $n \leq d$, an assumption $\theta_n \neq 0$ leads to a set of contradictory equations, and so θ_n must equal zero. The set of linear equations (8) then becomes a reduced set with the appropriate p_n/θ_n omitted, whilst (7) becomes

$$\sum_{n \in \mathcal{N}} \frac{r_n p_n}{\theta_n} = \frac{1}{a}$$

where $\mathcal{N} = \{n : p_n > 0,\ 1 \leq n \leq d+1\}$ and $r_n = 1 + \max\left\{k : \sum_{i=0}^{k} p_{n-i} = p_n\right\}$. Thus (9) still holds. On the other hand, if $p_{d+1} = 0$, the equations which replace (7) and (8) are always contradictory and so no invariant f of our postulated form exists.

From (9) and (5), it is seen that with

$$\beta_n = a\left[n - (d+1)\sum_{i=1}^{n} p_i\right] \tag{10}$$

an f of the form $\prod_{i=1}^{d} \beta_i x_i^{\beta_i - 1}$ is a solution of (4). Since in our context this f should also be a joint probability density function, it is necessary that $\beta_i > 0$. Hence f is an invariant probability density function of the Markov process of shapes provided

$$\sum_{i=1}^{n} p_i < \frac{n}{d+1}, \qquad 1 \leq n \leq d. \tag{11}$$

The Markov process is clearly aperiodic and ϕ-recurrent, in the sense of Orey (1971). Here ϕ can be Lebesgue measure on the state space $[0,1]^d$. Thus the invariant distribution we have found is unique and represents the equilibrium distribution of the process regardless of the initial state, provided of course that (11) holds.

Note that the non-existence of an invariant distribution when $p_{d+1} = 0$ is covered by (11), since (11) implies that $p_{d+1} > 1/(d+1)$. Whilst I have *not* proven that the shapes fail to have a non-degenerate limiting distribution when (11) is violated, I believe this to be the case. The methods of Mannion (1990, 1993), who proved limiting degeneracy in the case of Markov processes of triangular shapes, seem ideally suited for establishing the necessity of (11). This is deferred, however, to a later study since there appears to be interesting problems about the shapes of prisms in those dimensions which do not degenerate.

The condition (11) is intuitively agreeable since it highlights that if the *longest* edge is chosen with more than an equal share of probability and the shorter edges with less, degeneracy involving the collapse of the short edges to zero will not occur.

3. Analysis of the rule with an equally-likely planar cut

The sampling of a hyperplane in a manner such that the chosen hyperplane is uniformly distributed in $\mathcal{H}$, the set of all hyperplane which split the prism into two rectangular prisms, is the same as selecting a point T uniformly distributed on the set of all edges which emanate from O. This point defines a random variable $U = |OT|/L_j$ for the edge, j say, on which T happens to land. The probability $p_j = L_j/(L_1 + L_2 + \cdots + L_{d+1})$, where $L_{d+1} = 1$ and $L_j = Y_j Y_{j+1} \cdots Y_d$, whilst U is uniformly distributed.

The basic transformations from $(\boldsymbol{Y}, U)$ to $(\boldsymbol{X}, S)$ are the same as in Section 2, but the equations (2), (3) and (4) take modified forms. For $d = 1$,

$$f(x) = \int_x^1 \frac{xf(x/s)}{s+x}\frac{ds}{s} + \frac{1}{x}\int_0^x \frac{sf(s/x)}{s+x}ds + \int_0^1 \frac{sf(sx)}{1+s}ds \tag{12}$$

whilst for $d = 2$,

$$\begin{aligned} f(x,y) = &\int_x^1 \frac{xyf(x/s,y)}{s(s+sy+xy)}ds + \int_y^1 \frac{yf(sx,y/s)}{s+y+sxy}ds \\ &+ \int_{xy}^x \frac{yf(s/x,xy/s)}{s+xy+sy}ds + \int_0^{xy} \frac{sf(y,s/xy)}{x(xy+s+sy)}ds \\ &+ \int_0^y \frac{sf(xy,s/y)}{y+s+sxy}ds + \int_0^1 \frac{sf(x,sy)}{1+sy+sxy}ds. \end{aligned} \tag{13}$$

The general equation, analogous to (4), contains all the same terms as (4) except that the p_i's are relocated inside the integrals and are functions of the arguments of f. For example, the first integral in (4) involving p_3 becomes

$$\int_{x_3}^1 p_3(x_1, sx_2, \frac{x_3}{s}, \boldsymbol{x}_4^d) f(x_1, sx_2, \frac{x_3}{s}, \boldsymbol{x}_4^d) ds$$

where

$$p_3(y_1, y_2, y_3, \cdots, y_d) = \frac{y_3 y_4 \cdots y_d}{1 + y_d + y_d y_{d-1} + \cdots + y_1 y_2 \cdots y_d}.$$

Prior to solving the general case, it is an instructive exercise for the reader to show that $f(x) = 2/(1+x)^2$ is a solution of (12) and that

$$f(x,y) = \frac{12y}{(1+y+xy)^3} \tag{14}$$

is a solution of (13). In both cases, the stated solution is a probability density function. In completing these exercises, one notes that, when the respective solutions are substituted into the RHS's of (12) or (13), sets of integrals which together span the range $[0,1]$ attain common integrands and therefore merge into one integral over the range $[0,1]$. For example in (13), $\int_x^1$ merges with $\int_{xy}^x$ and with $\int_0^{xy}$. The merged expressions then comprise terms each expressible in the generic form

$$b\int_0^1 \frac{s^n\,ds}{(s+c)^{n+2}} = bI_n(c), \tag{15}$$

say, for some b and c (dependent on variables such as x and y) and with $n = d$. $I_n(c)$ evaluates as

$$I_n(c) = \frac{1}{(n+1)c(1+c)^{n+1}}. \tag{16}$$

In the general case, a similar merging of integrals into terms of the generic form (15) occurs when the postulated probability density (1) is used. One can then easily show that (1) provides a solution to the general equation.

4. Discussion of the first problem

In our first problem, the invariant distribution had independent components with marginal density functions belonging to the class $\mathcal{F} = \{f : f(x) = \gamma x^{\gamma-1};\ \gamma > 0\}$. Initially, my explorations of this problem focused on the case where $U \sim$ uniform, this seeming canonical, but since the solution involved $\mathcal{F}$, I was then motivated to investigate $U \sim g \in \mathcal{F}$ too. There was no added cost to the theory, though when I took the exercise further to the bigger class of Beta distributions, the algebraic costs were considerable. The powers of $(1-u)$ complicated matters greatly.

Such complicating terms also come in, even when $g \in \mathcal{F}$, if one investigates the equilibrium shape distribution of the discarded prism, the one *not* containing O. For example, when $d = 1$, the shape R of this (rejected) prism has pdf analogous to (12), with q for p_1,

$$f_R(r) = q\int_r^1 \frac{f(r/s)g(1-s)}{s}ds + (1-q)\left[\frac{1}{r^2}\int_0^r sf(s/r)g(1-s)ds + \int_0^1 sf(sr)g(1-s)ds\right]$$

where $g(u) = au^{a-1}$, $f(y) = \beta y^{\beta-1}$ and $\beta = a(1-2q)$. For general a and q, this expression involves incomplete Beta functions and Hypergeometric functions. When

$a = 1$, $f_R(r) = (1-2q)r^{-2q}$, obviously so since g is now symmetric and the discarded shape R has the same distribution as the retained one, X. For integer-valued $a \geq 2$, then

$$f_R(r) = \binom{a}{2}(1-2q)\Big\{r^{a-1-2aq}\sum_{i=1}^{a-1}(-1)^i\binom{a-2}{i-1}\Big(\frac{1}{2aq-a-i} - \frac{1}{2aq-a+i}\Big)$$
$$- \sum_{i=1}^{a-1}(-r)^{i-1}\binom{a-2}{i-1}\Big(\frac{1}{2aq-a-i} + \frac{1}{2aq-a+i}\Big)\Big\}$$

provided $q \neq (a-k)/2a$, $1 \leq k \leq a-1$. When $q = (a-k)/2a$

$$f_R(r) = \binom{a}{2}\frac{k}{a}\Big\{(-1)^k\binom{a-2}{k-1}r^{k-1}[\log r - \frac{1}{k}]$$
$$+ \sum_{\substack{i=1\\ i\neq k}}^{a-1}(-1)^i\binom{a-2}{i-1}\Big[\frac{(r^{k-1}-r^{i-1})}{k-i} - \frac{(r^{k-1}+r^{i-1})}{k+i}\Big]\Big\}.$$

In the symmetric case $a = 1$, the joint pdf of the identically-distributed variates X and R is interesting. For example, when $d = 1$,

$$\begin{aligned} f(x,r) &= (1-2q)\big\{q(r+x)^{-(2q+1)} + (1-q)A(x,r)\big\} \qquad && x+r<1,\\ &= (1-2q)(1-q)\big\{(r+x)^{2q-3} + A(x,r)\big\} && x+r>1, \end{aligned}$$

where

$$A(x,r) = \frac{r^{1-2q}+x^{1-2q}}{(1+rx)^{3-2q}} + \frac{(rx)^{1-2q}}{(x+r)^{3-2q}}.$$

When $q = 0$, that is when the long side of the rectangle is always the one chosen, X and R are both uniformly distributed with joint pdf on $[0,1]^2$

$$\begin{aligned} f(x,r) &= \frac{r+x}{(1+rx)^3} + \frac{rx}{(x+r)^3} \qquad && x+r<1\\ &= \frac{r+x}{(1+rx)^3} + \frac{1+rx}{(x+r)^3} && x+r>1. \end{aligned}$$

The correlation coefficient of X and R is $11 - 16\log(2) \doteqdot -0.0904$, in this case.

The d components of shape can be viewed as the shapes of two-dimensional rectangular 'faces'. This is easily visualised for a 3-dimensional prism where $d = 2$. Three edges, of lengths L_1, L_2 and L_3 emanate from O. These are the sides of rectangular faces $L_1 \times L_2$, $L_2 \times L_3$ and $L_1 \times L_3$. The first two of these have shapes X_1 and X_2

respectively. The third rectangular face has a shape too, namely X_1X_2. At equilibrium, $Z \equiv X_1X_2$ has pdf

$$f_Z(z) = \frac{a(1-3p_1)(3p_3-1)\left[z^{a(3p_3-1)} - z^{a(1-3p_1)}\right]}{(3p_2-1)z}.$$

5. Discussion of the second problem

When the cutting plane is an equally likely choice from $\mathcal{H}$, the marginal pdf of X_k is, for $d \geq k \geq 1$,

$$f_k(x) = \frac{(d+1)!x^{k-1}}{(d-k)!}\left[\prod_{i=1}^{k} \frac{1}{d+1-k+ix}\right]\left[\sum_{i=1}^{k} \frac{1}{d+1-k+ix}\right]$$

which interestingly converges to kx^{k-1} as $d \to \infty$ with k fixed. For example,

$$f_1(x) = \frac{(d+1)!}{(d-1)!}\frac{1}{(d+x)^2} \qquad d \geq 1.$$

When $d = 2$, the marginals for X_1 and X_2 are (using x and y),

$$f_1(x) = \frac{6}{(2+x)^2}, \qquad f_2(y) = \frac{6y(2+3y)}{(1+y)^2(1+2y)^2},$$

whilst the 'third rectangle' has shape Z with pdf

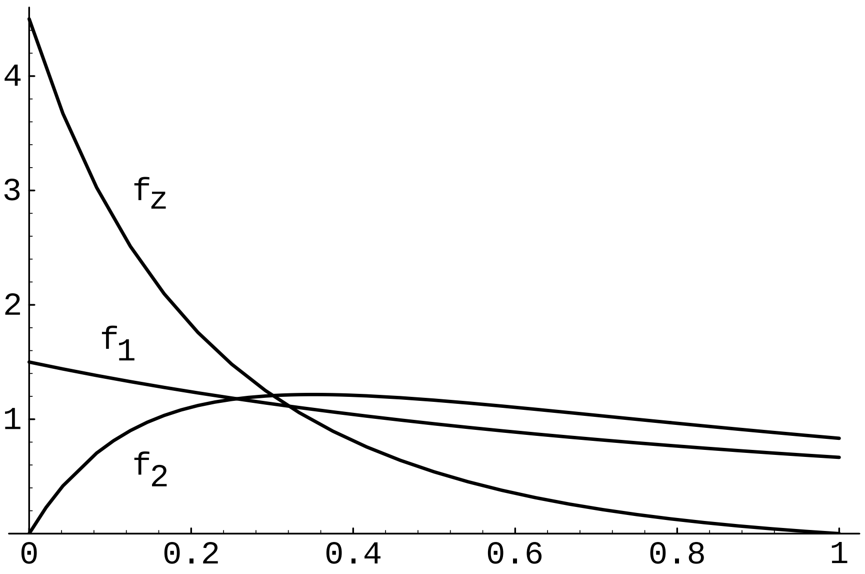
$$f_Z(z) = \frac{18(1-z^2)}{(2+z)^2(1+2z)^2}.$$

These three density functions, which are plotted in the figure below, are derived from (14). Note that $E(X) = 6\log(3/2) - 2 \doteqdot 0.4328$, $E(Y) = 6\log 2 - (3/2)\log 3 - 2 \doteqdot 0.5110$, $\mathrm{Var}(X) \doteqdot 0.0815$, $\mathrm{Var}(Y) \doteqdot 0.0691$, whilst $\rho_{XY} = -0.0801$. $E(Z) = 1 + 6\log 2 - (9/2)\log 3 \doteqdot 0.2151$. Note that derivations of these marginal densities exploit the identity (16).

When $d = 1$, the joint pdf of X and R on $[0,1]^2$ is

$$f(x,r) = \frac{2}{(1+r+x)^3} + \frac{2r}{(1+rx+r)^3} + \frac{2rx}{(x+r+rx)^3} + \frac{2x}{(1+rx+x)^3}.$$

X and R are identically distributed with mean $2\log 2 - 1$, variance $2 - 4(\log 2)^2$ and correlation coefficient 0.154.



References

Bookstein, F. (1978). *The measurement of biological shape and shape change*, Lecture Notes in Biomathematics **24**, Springer, New York..

Kendall, D. G. (1977). The diffusion of shape. *Adv. Appl. Prob.* **9**, 428-430..

Kendall, D. G. (1984). Shape manifolds, procrustean metrics and complex projective spaces. *Bull. Lond. Math. Soc.* **16**, 81-121..

Mannion, D. (1990). Convergence to collinearity of a sequence of random triangle shapes. *Adv. Appl. Prob.* **22**, 831–844.

Mannion, D. (1993). Products of 2×2 random matrices. *Annals Appl. Prob.* **3**, 1189–1218.

Miles, R.E. (1983). On the repeated splitting of a planar domain. *Proceedings of Oberwolfach conference on Stochastic Geometry, Geometric Statistics and Stereology.* 110-123 Eds. R. Ambartzumian & W. Weil Teubner, Leipzig.

Orey, S. (1971). *Limit theorems for Markov chain transition probabilities.* Van Nostrand Reinhold, London.

Watson, G. (1986). Random triangles. In *Proc. Conf. Stochastic Geometry.* Ed. O. Barndorff-Nielsen. Aarhus..