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INVARIANT DISTRIBUTIONS FOR SHAPES IN SEQUENCES OF RANDOMLY-DIVIDED RECTANGLES

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Abstract

Interest has been shown in Markovian sequences of geometric shapes. Mostly the equations for invariant probability measures over shape space are extremely complicated and multidimensional. This paper deals with rectangles which have a simple one-dimensional shape descriptor. We explore the invariant distributions of shape under a variety of randomised rules for splitting the rectangle into two sub-rectangles, with numerous methods for selecting the next shape in sequence. Many explicit results emerge. These help to fill a vacant niche in shape theory, whilst contributing at the same time, new distributions on [0,1] and interesting examples of Markov processes or, in the language of another discipline, of stochastic dynamical systems.

INVARIANT DISTRIBUTIONS; SHAPE; STOCHASTIC GEOMETRY; DISTRIBUTION THEORY; MARKOV PROCESSES; DYNAMICAL SYSTEMS.

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SECONDARY 60E05;60J05

1. Introduction

Many authors have addressed problems concerning the shapes of Markovian sequences of triangles (Watson [11], Mannion [7, 8], Kendall [4, 5], Miles [9]). Interest has focused on the limiting distribution of shape. In many cases, as Mannion has shown, the shapes degenerate since the three vertices become collinear with probability 1. In other cases, however, a non-degenerate limiting distribution exists.

The specification of shape for a triangle involves two quantities and so the equations for the invariant probability measure are two dimensional and usually rather difficult to solve. It is also difficult to completely characterise the conditions for degeneracy, although Mannion's work goes a long way towards this characterisation for shape sequences which can be generated by products of random matrices.

We have found it fruitful to explore shape in a context where only *one* quantity is required to define the shape of a geometric entity. We have noted that the shape of a rectangle in two-dimensions can be defined by a single number, for example, the ratio

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of the shorter to longer side. Using this definition, which is convenient in many ways but especially because shape then lies in $[0,1]$, we have studied numerous models for Markovian sequences of rectangles. In this paper, we focus on models where the rectangle is divided randomly by a line into two rectangular domains, one of which is selected as the next in sequence. We find a range of interesting limiting behaviour in these models and are able to give clear conditions to establish the borderline between degenerate and non-degenerate cases. The paper contributes partly to shape theory, partly to distribution theory and partly, via a repertoire of interesting examples, to the theories of Markov processes and dynamical systems.

In this paper we deal with sequences of shapes $X_0, X_1, X_2, \ldots$, these forming a discrete-time Markov chain with state space $[0,1]$. We find, in each of our case studies, $\mathbb{P}\{X_{n+1} \leq x|y\}$, where y is the realisation of X_n, and show the existence of the transition probability density function $p(\cdot|y)$. By evaluating $p(\cdot|y)$, we investigate solutions of the standard equation for an invariant density function f of shape, namely

$$f(x) = \int_0^1 p(x|y)f(y)dy. \tag{1}$$

2. Sequential splitting models: p-rules

Consider at a certain stage a rectangle R with one corner labelled O. With probability p a point T is located with uniform distribution on the *long* side of R emanating from O. With probability $q = 1 - p$, T is uniformly distributed on the *short* side emerging from O. A dividing line through T is then drawn orthogonal to OT, thereby splitting R into two 'daughter' rectangles. The daughter containing O (say) is chosen as the next rectangle in the sequence. For convenience we rescale the long side to be 1 after each split and place this side on the horizontal axis at each stage whenever Cartesian representations are needed. As part of a study of d-dimensional shapes, this model has been studied in Cowan [3] where it is shown that there is a non-degenerate equilibrium shape distribution if $p > 1/2$ and that this equilibrium density is $f(x) = (2p-1)x^{2p-2}$, $x \in (0,1]$. If $p \leq 1/2$, the rectangles became progressively more elongated; their shapes converge in probability to zero (Le and Cowan [6]). Interestingly when $p = 1$, the equilibrium shape is uniformly distributed on $[0,1]$.

We now consider different ways for selecting the retained daughter rectangle. We introduce the idea of a selection point S placed in the rectangle prior to its division. After division, the daughter containing S is selected. If S is distributed uniformly in the interior of the rectangle, independently at each stage, we have a model, called **model $[\boldsymbol{p},\boldsymbol{\mathcal{I}}]$** where '$\boldsymbol{\mathcal{I}}$' is for 'interior', shown in Figure 1. The '$\boldsymbol{p}$' distinguishes between the p-rule case of this section and another case considered later. (Bold font in this notation is used only when first mentioning a model, thus allowing easy back referral.) We also consider other randomisations for S, a model where S is uniformly distributed on the boundary of the rectangle, **model $[\boldsymbol{p},\boldsymbol{\mathcal{B}}]$**, and a model with S uniformly distributed at one of the four corners, **model $[\boldsymbol{p},\boldsymbol{\mathcal{C}}]$**.

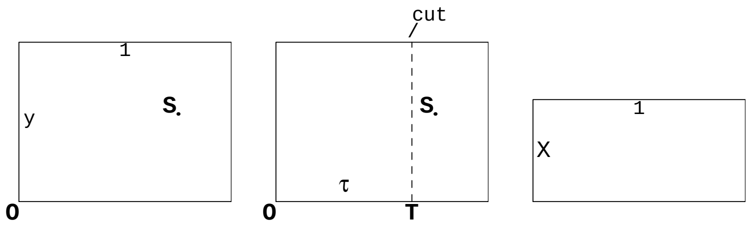


Figure 1: The operation of a selection point S for the model $[p, \mathcal{I}]$. The cut position T, shown here on the long side, is a proportion τ along the chosen side. Note that the long side is always placed horizontally, a useful convention when we later introduce the coordinates of S.

To establish $p(x|y)$ for all three of these models, we firstly find $\mathbb{P}\{X_{n+1} \le x|y\}$, where y is the realisation of X_n. Define τ as the proportion of OT to the length of the chosen side. Let $\mathcal{L}$ be the event that the long side is cut and $\mathcal{E}$ be the event that, after the cut is decided, S is placed in the same daughter as O. Writing X for X_{n+1},

$$\begin{aligned}\mathbb{P}\{X \le x|y\} &= p\mathbb{P}\{X \le x|y, \mathcal{L}\} + q\mathbb{P}\{X \le x|y, \bar{\mathcal{L}}\} \\ &= p\int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau\}d\tau + q\int_0^1 \mathbb{P}\{X \le x|y, \bar{\mathcal{L}}, \tau\}d\tau.\end{aligned}$$

Focussing on the first integral,

$$\begin{aligned}&\int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau\}d\tau \\ &= \int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \mathcal{L}, \tau\}d\tau + \int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau, \bar{\mathcal{E}}\}\mathbb{P}\{\bar{\mathcal{E}}|y, \mathcal{L}, \tau\}d\tau \\ &= \int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \mathcal{L}, \tau\}d\tau \\ &\qquad + \int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, 1-\tau, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \mathcal{L}, 1-\tau\}d\tau \\ &= \int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \mathcal{L}, \tau\}d\tau + \int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \eta, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \mathcal{L}, \eta\}d\eta \\ &= 2\int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \mathcal{L}, \tau\}d\tau.\end{aligned}$$

In the first step above, we simply note that the event $\bar{\mathcal{E}}$ with reference to τ becomes the event $\mathcal{E}$ with reference to $1-\tau$. (Just envisage O relocated to the other end of the chosen side and note that $1-\tau$ is uniformly distributed too.) The second step is the formal change of variable, $\eta = 1-\tau$. Thus

$$\begin{aligned}\mathbb{P}\{X \le x|y\} &= 2p\int_0^1 \mathbb{P}\{X \le x|y, \mathcal{L}, \tau, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \mathcal{L}, \tau\}d\tau \\ &\qquad + 2q\int_0^1 \mathbb{P}\{X \le x|y, \bar{\mathcal{L}}, \tau, \mathcal{E}\}\mathbb{P}\{\mathcal{E}|y, \bar{\mathcal{L}}, \tau\}d\tau \\ &= 2(pI + qJ) \qquad \text{say,}\end{aligned}$$

where a similar argument has been used for the second term. For our three cases, one can easily consider the randomisation of S and conclude that

$$\begin{aligned}\mathbb{P}\{\mathcal{E}|y,\mathcal{L},\tau\} &= \tau && \text{model}[p,\mathcal{I}]\\ &= \frac{2\tau+y}{2(1+y)} && \text{model}[p,\mathcal{B}]\\ &= \tfrac{1}{2} && \text{model}[p,\mathcal{C}],\end{aligned}$$

whilst $\mathbb{P}\{\mathcal{E}|y,\bar{\mathcal{L}},\tau\}$ takes the same form as above under $[p,\mathcal{I}]$ and $[p,\mathcal{C}]$, but equals $(2\tau y+1)/(2(1+y))$ under $[p,\mathcal{B}]$. So, for model $[p,\mathcal{I}]$, one can show that, for $x<y$,

$$\begin{aligned}I &= \int_0^{xy} \tau d\tau = \tfrac{1}{2}x^2y^2\\ J &= \int_0^{x/y} \tau d\tau = \tfrac{1}{2}x^2/y^2\end{aligned}$$

and, for $x>y$,

$$\begin{aligned}I &= \int_0^{xy} \tau d\tau + \int_{y/x}^{1} \tau d\tau = \tfrac{1}{2}(x^2y^2+1-y^2/x^2)\\ J &= \int_0^{1} \tau d\tau = \tfrac{1}{2}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{P}\{X\le x|y\} &= px^2y^2+q\frac{x^2}{y^2} && 0\le x<y\\ &= p\big(x^2y^2+1-\frac{y^2}{x^2}\big)+q && y\le x\le 1\\ p(x|y) &= 2x\big(py^2+\frac{q}{y^2}\big) && 0\le x<y\\ &= 2py^2\big(x+\frac{1}{x^3}\big) && y<x\le 1.\end{aligned}$$

Therefore (1) takes the form

$$f(x) = \int_0^x 2py^2\big(x+\frac{1}{x^3}\big)f(y)dy + \int_x^1 2x\big(py^2+\frac{q}{y^2}\big)f(y)dy. \tag{2}$$

Differentiation twice converts (2) into a differential equation of Euler's type,

$$x^2f''(x)+(5-4p)xf'(x)+(3-4p)f(x)=0.$$

So this differential equation has solutions $f(x)=Ax^{4p-3}+Bx^{-1}$. In order that f is a probability density function integrating to 1 over $[0,1]$, we must set $B=0$ and also specify that $4p-3>-1$, that is, $p>\frac{1}{2}$. Thus $A=4p-2$ and we have the final solution, for $p>\frac{1}{2}$,

$$f(x)=(4p-2)x^{4p-3} \qquad x\in(0,1]. \tag{3}$$

A check confirms that this satisfies (2) and is always non-negative. As before, when $p \le \frac{1}{2}$ the shape converges in probability to zero.

Integrals I and J can readily be evaluated for model $[p, \mathcal{B}]$, yielding an absolutely continuous conditional distribution function whose density is

$$\begin{aligned} p(x|y) &= \frac{(2x+1)}{1+y}\left(py^2 + \frac{q}{y}\right) && 0 \le x < y \\ &= \frac{py^2}{1+y}\left(2x + 1 + \frac{2}{x^3} + \frac{1}{x^2}\right) && y < x \le 1. \end{aligned}$$

The resulting integral equation has a unique solution in the class of probability density functions provided $p > 2/3$.

$$f(x) = \frac{(3p-1)(3p-2)}{3(2p-1)}(1+x)x^{3p-3}. \tag{4}$$

Formal calculation for model $[p, \mathcal{C}]$ using the methods above shows that $[p, \mathcal{C}]$ is equivalent to the Cowan model, as one might expect.

Figure 2 shows the three main results to date, probability densities for a range of p in three models. Missing detail in the foregoing (and subsequent) analyses can be found in Chen and Cowan [2].

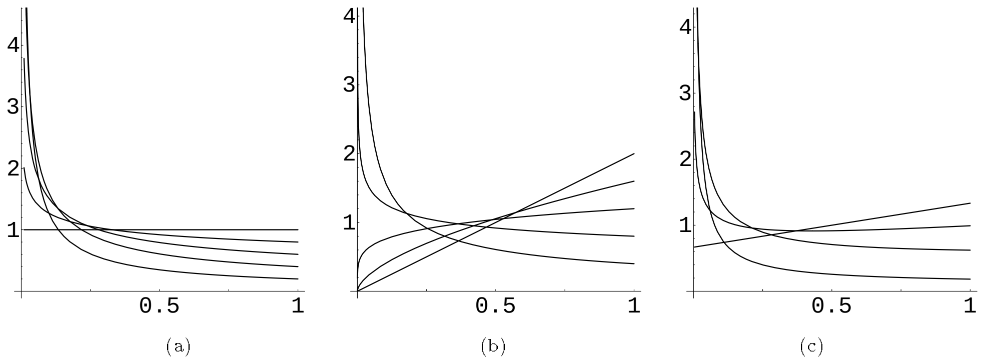


Figure 2: Invariant probability densities for: (a) Cowan's model for $p \in \{0.6, 0.7, 0.8, 0.9, 1.0\}$; (b) model $[p, \mathcal{I}]$ for similar p; (c) model $[p, \mathcal{B}]$ for $p \in \{0.7, 0.8, 0.9, 1.0\}$. Curves which are higher at $x = 1$ have higher p-value.

In trying to explain why the condition on p in model $[p, \mathcal{B}]$ is more strict, we focus on the case where y is very small, corresponding to a critically thin rectangle. Then $[p, \mathcal{B}]$ has a similar bias to $[p, \mathcal{I}]$ if T falls on the long side (both having a bias toward selection of the thinner of the daughters). On the other hand if T falls on the short side, the $[p, \mathcal{B}]$ model is shape neutral in its selection of the daughter, whilst $[p, \mathcal{I}]$ tends to select the daughter of larger area and hence larger shape. Thus $[p, \mathcal{B}]$ gives inferior recovery chances from the critically thin case and so a larger p is needed as compensation. It can also be argued for small y that $[p, \mathcal{B}]$ has inferior recovery compared with $[p, \mathcal{C}]$, since they both have shape neutral selection when T is on the short side whilst $[p,\mathcal{C}]$ does not have a bias toward the thinner shaped daughter when T is on the long side.

It can be useful to express the problems above in the language of stochastic dynamical systems (Arnold and Crauel [1]), where the focus is on the mapping which is iteratively applied. Our maps are of the form $X_{n+1} = g[X_n, \tau, \xi, \psi]$ where $(\xi, \psi y)$ are the coordinates of S. Defining $\chi(\cdot)$ as the indicator function,

$$
\begin{aligned}
X_{n+1} &= \chi(\mathcal{L}\cap\mathcal{E})\min(\frac{X_n}{\tau}, \frac{\tau}{X_n}) + \chi(\mathcal{L}\cap\bar{\mathcal{E}})\min(\frac{X_n}{1-\tau}, \frac{1-\tau}{X_n}) \\
&\quad + \chi(\bar{\mathcal{L}}\cap\mathcal{E})\tau X_n + \chi(\bar{\mathcal{L}}\cap\bar{\mathcal{E}})(1-\tau)X_n. \qquad (5)
\end{aligned}
$$

A briefer and more edifying form than (5) can be written using just two random variables, ρ and κ say, which capture the joint effects of τ, ξ and ψ.

$$
X_{n+1} = \chi(\mathcal{L})\min(\frac{X_n}{\rho}, \frac{\rho}{X_n}) + \chi(\bar{\mathcal{L}})\kappa X_n. \qquad (6)
$$

Here $\chi(\mathcal{L})$ is a Bernoulli(p) variate. Given X_n, ρ and κ are independent variates distributed like τ conditional upon, respectively, $\mathcal{L}\cap\mathcal{E}$ or $\bar{\mathcal{L}}\cap\mathcal{E}$. These two conditioning events, which may depend on X_n, are $\{\xi < \tau\}$ and $\{\psi < \tau\}$, respectively. So, for example, writing y for X_n,

$$
\begin{aligned}
\mathbb{P}\{\rho \le x|y\} &= \mathbb{P}\{\tau \le x|y, \xi < \tau\} \\
&= x^2 && \text{model}[p,\mathcal{I}] \\
&= \frac{x(x+y)}{1+y} && \text{model}[p,\mathcal{B}] \qquad (7)\\
&= x && \text{model}[p,\mathcal{C}],
\end{aligned}
$$

whilst κ has the same distribution function except for the $[p,\mathcal{B}]$ case, where it equals $x(1+xy)/(1+y)$. Note the dependence on y in the $[p,\mathcal{B}]$ case. The mapping (6) gives a further unity to the three problems and, indeed, to those that follow. In general, ρ and κ are distinct independent variates, but one notes that, in $[p,\mathcal{I}]$ and $[p,\mathcal{C}]$, one could replace κ in the second term by ρ. Note that the Greek-letter variables introduced above should really be subscripted with n, as should the events $\mathcal{L}$ and $\mathcal{E}$. We have avoided the clutter of this, asking the reader to envisage independent realisations at each stage.

Given that the cutting position T is randomly located via a p-rule mechanism, it seems natural to investigate the case (**model $[p,p']$**) where S is similarly randomised on the boundary, with a probability p' of being located uniformly on the long side and a probability $q' = 1-p'$ of being uniformly distributed on the short side. We find (details omitted) that the density $p(\cdot|y)$ exists, with

$$
\begin{aligned}
p(x|y) &= 2x\big(pp'y^2 + \frac{qq'}{y^2}\big) + pq'y + \frac{qp'}{y} && 0 \le x < y \\
&= 2pp'y^2\big(x + \frac{1}{x^3}\big) + pq'y\big(1 + \frac{1}{x^2}\big) && y < x \le 1.
\end{aligned}
$$

A lengthy analysis of the resulting integral equation shows that a unique invariant pdf exists provided $3p > 1 + p'$, a condition explained by arguments about critically thin rectangles (not detailed here but similar to those used earlier). Its form is $f(x) = Ax^{\alpha} + Bx^{\beta}$ where α and β are the two real roots of the cubic equation

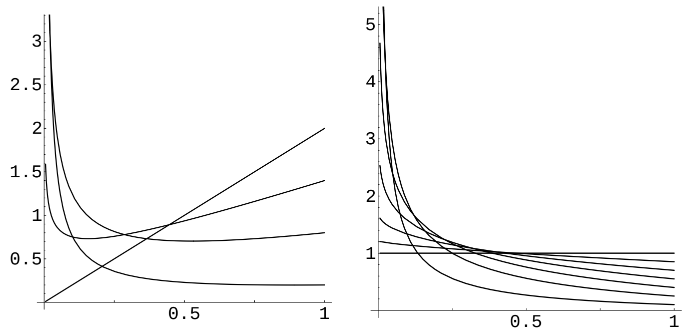
$$r^3 + (2 + 3q - p')r^2 + (p' + 9q - 3 - 6p'q)r - 6p'q = 0$$

which lie in $(-1, \infty)$. We have proved that there exist two such roots when $3p > 1+p'$. The coefficients A and B are the unique solutions of

$$\begin{aligned}
&[\frac{pq'}{\alpha+2} + \frac{qp'}{\alpha}]A + [\frac{pq'}{\beta+2} + \frac{qp'}{\beta}]B = 0 \\
&\qquad\qquad (\beta+1)A + (\alpha+1)B = (\alpha+1)(\beta+1).
\end{aligned}$$

There are noteworthy special cases, listed below and plotted in Figure 3:

$$\begin{aligned}
f(x) &= \frac{2(3p-2)}{4-3p}\big[x + 3(1-p)x^{3p-3}\big] && p' = 1,\ p > \tfrac{2}{3}; \\
f(x) &= \frac{3p-1}{6p-4}\big[1 - 3(1-p)x^{3p-2}\big] && p' = 0,\ p > \tfrac{1}{3}; \\
f(x) &= \frac{2(1-p'+p'x)}{2-p'} && p = 1.
\end{aligned}$$



(a) (b)

Figure 3: Invariant probability densities for model $[p, p']$. (a) $p' = 1$ and $p \in \{0.7, 0.8, 0.9, 1.0\}$; (b) $p' = 0$ and $p \in \{0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$. Curves with higher values at $x = 1$ have higher p-values.

Models $[p, \mathcal{I}]$ and $[p, \mathcal{B}]$ have biases toward 'big' daughter rectangles. The former biases toward the selection of daughters with large area, the latter, though not purely perimeter biasing, tends to give added weight on the basis of 'perimeter'. To have 'pure perimeter biassing', one needs to distribute S uniformly on the union of the post-separation boundaries of the two daughters (**model $[\boldsymbol{p}, \boldsymbol{\mathcal{P}}]$**), that is, with the division line being counted twice as a side for each daughter. In a manner similar to model $[p, \mathcal{B}]$ we find that $p > 2/3$ is a requirement for an invariant distribution. Its form is

$$f(x) = \frac{A_p(1+x)^2x^{3p-3}}{(2+x)^{1-p}(1+2x)^p}$$

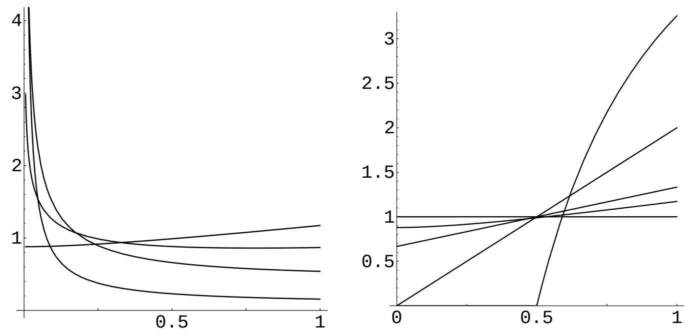
where the constant A_p needs to be evaluated numerically except when $p = 1$, whence $A_1 = 8/(8 + \log 3)$. Plots are shown in Figure 4(a).

Area and perimeter are not themselves indicators of shape, so we look directly at the selection rule (**model** $[\boldsymbol{p}]_{\mathbf{FAT}}$) where the 'fatter' daughter, that is, the one with larger shape, is *always* the one selected. Thus we have abandoned the notion of a selection point S here. We have solved this problem only when $p = 1$, whereupon

$$\begin{aligned} f(x) &= 0 & x &< \tfrac{1}{2} \\ &= \frac{2 - 1/x}{1 - \log 2} & x &\geq \tfrac{1}{2}. \end{aligned} \tag{8}$$

Our attempts to analyse the case where the 'thinner' daughter is always selected have been unsuccessful.

Comparative plots when $p = 1$ for all relevant models to date are shown in Figure 4(b). The biases mentioned above are evident. The selection of the fatter daughter understandably leads to the greatest bias compared with the Cowan model, $[p, \mathcal{C}]$, but the area and perimeter biasing of models $[p, \mathcal{I}]$, $[p, \mathcal{B}]$ and $[p, \mathcal{P}]$ create some bias in shape.



(a) (b)

Figure 4: Invariant probability density functions for (a) model $[p, \mathcal{P}]$ when $p \in \{0.7, 0.8, 0.9, 1.0\}$, curves higher at $x = 1$ having higher p values, and (b) for most models to date with $p = 1$, where the curves in ascending order at $x = 1$ are $\mathcal{C}$, $\mathcal{P}$, $\mathcal{B}$, $\mathcal{I}$, FAT.

As dynamical systems, the last three cases are also governed by (6), with the distributions for ρ and κ as follows (derivation omitted).

$$\begin{aligned} \mathbb{P}\{\rho \leq x|y\} &= x(q' + p'x) & &\text{model}[p, p'] \\ &= \frac{x(x + 2y)}{1 + 2y} & &\text{model}[p, \mathcal{P}] \\ \mathbb{P}\{\kappa \leq x|y\} &= \frac{x(p' + q'xy)}{p' + q'y} & &\text{model}[p, p'] \\ &= \frac{x(2 + xy)}{2 + y} & &\text{model}[p, \mathcal{P}], \end{aligned} \tag{9}$$

whilst, under model $[p]_{FAT}$ for $p = 1$, defining u as $(1 - \sqrt{1 - 4y^2})/2$,

$$
\begin{aligned}
\mathbb{P}\{\rho \le x|y\} &= 2(x-u)\chi(x \in [u, \tfrac{1}{2}]) + (1-2u)\chi(x \in (\tfrac{1}{2}, 1-u)) \\
&\quad + (2x-1)\chi(x \in [1-u, 1]) && y < \tfrac{1}{2} \\
&= (2x-1)\chi(x > \tfrac{1}{2}) && y \ge \tfrac{1}{2} \\
\mathbb{P}\{\kappa \le x|y\} &= (2x-1)\chi(x > \tfrac{1}{2}).
\end{aligned} \tag{10}
$$

3. p-models with fixed selection point S

At each stage of the splitting process, the rectangle is drawn to have its long side, scaled to 1, on the x-axis. A point S whose coordinates are $(c_1, c_2 y)$ is placed on the figure, $0 \le c_1 \le \frac{1}{2}$, $0 \le c_2 \le \frac{1}{2}$. The daughter containing S is selected, but in this version (**model [*p*,relatively fixed]**) c_1 and c_2 are fixed throughout the process. Thus S is fixed in a proportional way. The case $(c_1, c_2) = (0, 0)$ is the Cowan model, $[p, \mathcal{C}]$.

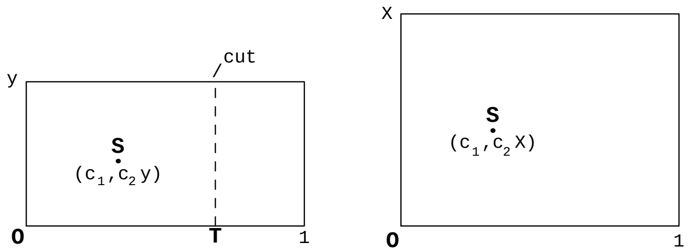


Figure 5: Two rectangles in the sequence showing the position of S.

We have not solved this problem for general (c_1, c_2) but have interesting special cases. In the case $c_2 = 0$, $0 < c_1 = c \le \frac{1}{2}$, standard though lengthy calculations lead to the following integral equations.

$$
f(x) = \begin{cases}
H(x,f) & \text{for } \ 0 \le x \le c, \\
H(x,f) + p\int_{c/x}^{1} yf(y)dy & \text{for } \ c \le x \le 1-c, \\
H(x,f) + p\int_{c/x}^{1} yf(y)dy + p\int_{(1-c)/x}^{1} yf(y)dy & \text{for } \ 1-c \le x \le 1,
\end{cases}
$$

where

$$
H(x,f) = \frac{p}{x^2}\left\{\int_{cx}^{x} yf(y)dy + \int_{(1-c)x}^{x} yf(y)dy\right\} + q\int_{x}^{1}\frac{f(y)}{y}dy.
$$

Introducing $f(x) = u(x), v(x)$ or $w(x)$ respectively in the ranges $[0, c], [c, 1-c]$ and $[1-c, 1]$, noting continuity of f at the joins, we can rewrite these equations in terms of u, v and w. We may write, for $0 \le x \le c$, an equation for u

$$
u(x) = \frac{p}{x^2}\left\{\int_{cx}^{x} yu(y)dy + \int_{(1-c)x}^{x} yu(y)dy\right\} + q\int_{x}^{c}\frac{u(y)}{y}dy + qQ
$$

where Q is a constant. When $p = 1$, $u(x) = 0$ is the unique solution amongst functions for which $\int_0^c u(y)dy < \infty$. For general $p < 1$, the solution is of the form $u(x) = Ax^\alpha$ where α is any solution in the range $(-1, \infty)$ of the equation

$$3p - 1 - \frac{4p}{\alpha+2} - \frac{\alpha p c^{\alpha+2}}{\alpha+2} - \frac{\alpha p(1-c)^{\alpha+2}}{\alpha+2} - \alpha = 0 \tag{11}$$

and A is a positive constant determined later. Analysis shows that there is a unique root of (11) greater than -1, and actually lying in $(-1, 0)$, iff

$$\frac{1}{2 + c\log c + (1-c)\log(1-c)} < p < 1. \tag{12}$$

Having solved for u in $[0, c]$, we can now write an equation for v in the range $[c, 1-c]$. In similar fashion, we can identify that, when p satisfies (11), v is of the form

$$v(x) = A\left[\frac{2pc}{Bx} + \frac{p\alpha c^{\alpha+2}}{Bx^{\alpha+2}} + \left\{1 - \frac{p(\alpha+2)}{Bc^\alpha}\right\}x^\alpha\right]$$

where α is the unique root of (11) and

$$B = p(\alpha+2)\{c^{-\alpha} + (1-c)^{-\alpha}\} - \alpha(\alpha + 3p + 1) - 4p.$$

Similarly, having solutions for u and v, we can establish an equation for w in $[1-c, 1]$. The solution is

$$w(x) = A\left[\frac{2p}{Bx} + \frac{\alpha p[c^{\alpha+2} + (1-c)^{\alpha+2}]}{Bx^{\alpha+2}} + \left\{1 - \frac{p(\alpha+2)[c^{-\alpha} + (1-c)^{-\alpha}]}{B}\right\}x^\alpha\right].$$

Finally, the constant A is found from the requirement that f integrates to unity over $[0, 1]$. This leads to

$$A = \frac{B}{2[1 - 2p - pc\log c - p(1-c)\log(1-c)]},$$

a constant which is positive when (11) is satisfied.

When $p = 1$, the solutions for v and w are somewhat simpler. In summary for this case, $u(x) = 0$,

$$\begin{aligned} v(x) &= \frac{1 - c/x}{1 + c\log c + (1-c)\log(1-c)} \qquad c \le x \le 1-c \\ w(x) &= \frac{2 - 1/x}{1 + c\log c + (1-c)\log(1-c)} \qquad 1-c \le x \le 1. \end{aligned}$$

The resulting pdf f is plotted in Figure 6(b) for a range of c. Strangely, the $p = 1$ solution for $c = 1/2$, agrees with (8). When $c = 1/2$, the daughter with simultaneously larger *area* and *perimeter* is always selected, whilst (8) gives the results when one selects the daughter with larger *shape*.

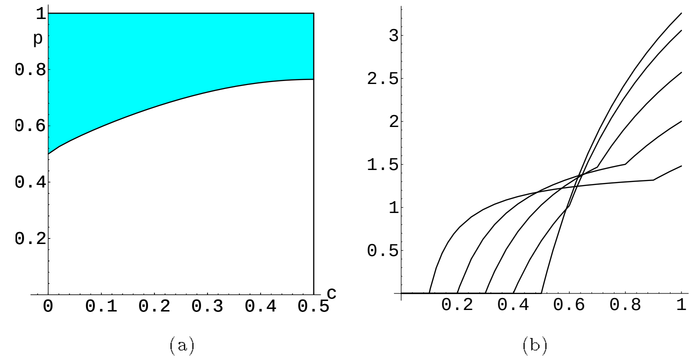


Figure 6: (a) The shaded region shows the valid combinations of (p, c) from (12). (b) The invariant f when $p = 1$ for a range of c with $c_2 = 0$.

The lower bound in (12) is $1/2$ when $c = 0$, rising monotonically to about 0.765 when $c = 1/2$. Thus the conditions for a non-degenerate equilibrium are more stringent as c increases (Figure 6(a)). To see why this is so intuitively, one need only consider a critically thin rectangle. When divided on the long side, with $c = 1/2$ say for illustration, the rectangle selected will usually be the thinner of the two daughters, whereas when $c = 0$, it is equally likely to be the thinner or thicker shaped. A positive c gives, for any p, a mechanism with less 'recovery' from critically thin rectangles. Thus a higher p is needed as compensation.

If one prefers to use the dynamical system map (6), then ρ has the uniform distribution on $[c_1, 1]$; κ is uniformly distributed on $[c_2, 1]$.

Instead of fixing S *relatively* via (c_1, c_2), we might consider the case where we commence with a given rectangle R_0 in the positive quadrant of the plane having one vertex at the planar origin. Now S is a point, with fixed planar coordinates, within R_0.

Figure 7 shows the early part of a realisation of $\{R_n\}$ for a fixed S and randomly located cuts determined by T. We have considered the (p, q) selection of T on long-short sides as before.

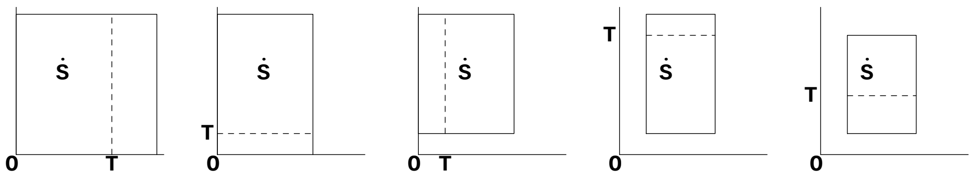


Figure 7: A sequence of rectangles with S fixed in space. Note that the point O is not the origin of the plane; it is the lower-left corner of the rectangle.

The 'fixed' nature of S is somewhat deceptive since its relative position within the subset $R_n \subset R_0$ is random. If S is an interior point of R_0 (**model** $[\boldsymbol{p}, \boldsymbol{\mathcal{I}}]_{\boldsymbol{fix}}$), then it is easy to show, that as $n \to \infty$, the relative position (ξ_n, η_n) of S within R_n becomes uniformly distributed in $[0, 1]^2$. Here ξ_n and η_n measure the proportions along the long and short sides respectively. $\{(\xi_n, \eta_n)\}_{n \geq 0}$ is a ϕ-recurrent Markov process. If (ξ_n, η_n) is uniformly distributed, then given that this point lies within any subset of R_n, in particular R_{n+1}, it is uniformly distributed within that subset. Thus the uniform distribution is an invariant one, and hence the limiting form due to the ϕ-recurrence.

So we might anticipate that the shape dynamics of $\{R_n\}$ is identical to model $[p, \mathcal{I}]$ with equilibrium pdf given by (3). Indeed $\{X_n\}$ is defined by (5), but now the successive values of (ξ_n, η_n) are dependent. Nevertheless, the augmented Markov process $\{(X_n, \xi_n, \eta_n)\}$ does have an invariant pdf given as the RHS of (3) on $[0,1]^3$.

If S is fixed on the boundary but not at a corner (**model $[p, \mathcal{B}]_{fix}$**), S becomes uniformly distributed on the side on which it started, using a similar argument. Call the position on this side at stage n, ζ_n. Given the event, $\mathcal{A}_n$, that S is on the long side at stage n, $\{X_n\}$ is defined by (5) with $\xi = \zeta$ and $\eta = 0$. Under $\bar{\mathcal{A}}_n$, (5) holds with $\mathcal{L}$ and $\bar{\mathcal{L}}$ interchanged and now $\eta = \zeta$, with $\xi = 0$. In both these versions, the ζ variates are sequentially dependent. Our analysis has shown that if the ζ variates are uniformly distributed at some stage, then the bivariate Markov process $\{(X_n, \chi(\mathcal{A}_n))\}$ has an invariant distribution provided $p > 2/3$, with the pdf for X being given by (4) and the probability of $\mathcal{A}$ being $(3p-1)/(3(2p-1))$, independent of X. Thus the tri-variate Markov process $\{(X_n, \chi(\mathcal{A}_n), \zeta_n)\}$, has the obvious invariant distribution involving shapes distributed like (4). Hence the limiting shape distribution agrees with the $[p, \mathcal{B}]$ model.

If S is a corner of R_0, it will always be a corner of R_n and so model $[p, \mathcal{C}]$ applies.

4. Sequential splitting rules: $\mathcal{H}$-rules

At each stage of the process, let $\mathcal{H}$ be the set of all lines which cut the rectangle into two rectangular daughters. Clearly these lines are orthogonal to either the long or short side. We consider an important randomisation where each line in $\mathcal{H}$ is equally likely to be the cutting line. This is equivalent to the placing of T uniformly distributed on the boundary of R_n. So the probability p that the long side is cut is dependent on the shape y of the rectangle, namely $p = 1/(1+y)$.

The process starts with one arbitrary corner being labelled O and, at each stage, the daughter rectangle including O (or equivalently a randomly-selected corner) is selected (**model $[\mathcal{H}, \mathcal{C}]$**). Cowan [3] has found the equilibrium distribution of shape, as part of a study on $(d+1)$-dimensional rectangular prisms. The equilibrium density is $f(x) = 2/(1+x)^2$. The 2-dimensional problem was considered earlier by Miles [9], but he used a different definition of shape.He placed the initial rectangle in the Cartesian plane with sides parallel to the planar axes and defined shape σ as the ratio of vertical to horizontal side lengths; daughter rectangles were also fixed in the plane with unchanged orientations. His equilibrium density, is $1/(1+\sigma)^2$ for $\sigma \in [0, \infty)$. The two results are equivalent, but we find that our definition of shape, as a quantity in the interval $[0,1]$, is more convenient.

As in Section 2, we can consider biases which arise due to the random placement of a selection point S. If S is uniformly distributed inside R_n and the daughter containing S is selected (**model** $[\boldsymbol{\mathcal{H}},\ \boldsymbol{\mathcal{I}}]$), then

$$\begin{aligned} p(x|y) &= \frac{2x}{1+y}\big(y^2+\frac{1}{y}\big) \qquad 0 \le x < y \\ &= \frac{2y^2}{1+y}\big(x+\frac{1}{x^3}\big) \qquad y \le x \le 1. \end{aligned}$$

The resulting equation is, from (1),

$$f(x) = 2\big(x+\frac{1}{x^3}\big)\int_0^x \frac{y^2 f(y)}{1+y}dy + 2x\int_x^1 \frac{(y^3+1)f(y)}{y(1+y)}dy$$

which has the unique pdf solution

$$f(x) = \frac{12x}{(1+x)^4}.$$

When S is uniformly distributed on the boundary of R_n (**model** $[\boldsymbol{\mathcal{H}},\ \boldsymbol{\mathcal{B}}]$), a similar analysis yields the invariant pdf as

$$f(x) = \frac{2}{(1+x)^2}, \tag{13}$$

interestingly the same invariant density function as in the Cowan/Miles case (model $[\mathcal{H},\ \mathcal{C}]$). We note, however, that the transition probabilities in the two cases differ.

When T follows the $\mathcal{H}$-rules and S is uniformly distributed on either the long or short side with probabilities p' and q', we have the dual of model $[p, \mathcal{B}]$, namely **model** $[\boldsymbol{\mathcal{H}}, \boldsymbol{p'}]$. We have found the invariant density in a simple form for the special cases $p' = 0$ or 1. These are:

$$\begin{aligned} f(x) &= \frac{2}{15}\left(1+x+\frac{16}{(1+x)^3}\right) && p' = 1; \\ f(x) &= \frac{2(2+21x-6x^2-x^3-6x\log x)}{7(1+x)^3} && p' = 0. \end{aligned} \tag{14}$$

The invariant density exists for all p' but is extremely complicated if our attempts for $p' = \frac{1}{2}$, which yielded a very lengthy and obfuscating expression, are any indication.

When S is such that pure 'perimeter biassing' occurs (**model** $[\boldsymbol{\mathcal{H}},\ \boldsymbol{\mathcal{P}}]$), that is, when S is uniformly distributed on the union of the daughters' post-separation boundaries, we find that

$$f(x) = \frac{54(1+x)^2}{(3+4\log 2)(2+x)^2(1+2x)^2}. \tag{15}$$

The models where selection of the fatter or thinner daughter occurs have proved intractable.

We have also considered fixed selection points. The model where S is fixed in a relative way within the rectangle has proved intractable, but the cases, **model**$[\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{I}}]_{\boldsymbol{fix}}$, **model**$[\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{B}}]_{\boldsymbol{fix}}$ and **model**$[\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{C}}]_{\boldsymbol{fix}}$, where S is fixed absolutely in Cartesian space, turn out, after similar considerations to that for the p-rule situation, to have invariant distributions like their respective random-S cases. For example, model $[\mathcal{H}, \mathcal{B}]_{fix}$, has the invariant pdf of (13), with the additional result that the side containing S is long with probability 3/4.

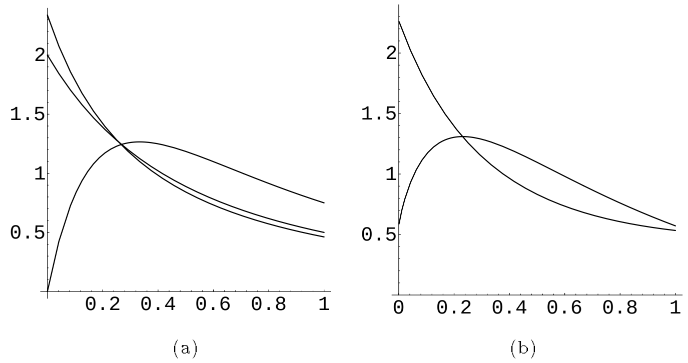


Figure 8: Plots for the $\mathcal{H}$-rule cases in ascending order at $x = 1$: (a) models $[\mathcal{H}, \mathcal{P}]$, $[\mathcal{H}, \mathcal{C}$ or $\mathcal{B}]$ and $[\mathcal{H}, \mathcal{I}]$; (b) model $[\mathcal{H}, p']$ for $p' = 1$ and 0.

The dynamical system for the $\mathcal{H}$-rule cases follows the same mapping (6) as for the p-rule cases, except that $\chi(\mathcal{L})$ is now a Bernoulli$[1/(1+y)]$ variate. The distributions for ρ and κ follow, as before, equations (7), (9) and (10) as appropriate.

5. Concluding remarks

Our effort has focused on finding invariant distributions and so we have not dwelt on proofs of ϕ-recurrence for each case, though it is clear that proofs could be developed using $\phi :=$ Lebesgue measure on [0,1]. This condition (Orey [10]), combined with obvious aperiodicity, establishes uniqueness of the invariant distributions.

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References

[1] Arnold, L. and Crauel, H. (1991). Random dynamical systems. In *Lyapunov Exponents*. Eds. L. Arnold, H. Crauel & J.-P. Eckmann. Lecture Notes in Mathematics 1486, Springer, Berlin.

[2] Chen, F. K. C. and Cowan, R. (1997). Proofs of Invariant Distributions for Shapes in Sequences of Randomly-divided Rectangles. *Research Report* **166**, Department of Statistics, University of Hong Kong (available at *http://www.maths.usyd.edu.au:8000/u/richardc/*).

[3] Cowan, R. (1997). Shapes of rectangular prisms after repeated random division. *Adv. Appl. Prob* **29**, 26–37.

[4] KENDALL, D. G. (1977). The diffusion of shape. *Adv. Appl. Prob.* **9,** 428–430.

[5] KENDALL, D. G. (1984). Shape manifolds, procrustean metrics and complex projective spaces. *Bull. Lond. Math. Soc.* **16,** 81–121.

[6] LE, H. AND COWAN, R. (1997). A proof, based on stochastic ordering, for degenerate convergence of shapes. *Research Report* **148,** Department of Statistics, University of Hong Kong.

[7] MANNION, D. (1990). Convergence to collinearity of a sequence of random triangle shapes. *Adv. Appl. Prob.* **22,** 831–844.

[8] MANNION, D. (1993). Products of 2×2 random matrices. *Ann. Appl. Prob.* **3,** 1189–1218.

[9] MILES, R. E. (1983). On the repeated splitting of a planar domain. *Proceedings of Oberwolfach conference on Stochastic Geometry, Geometric Statistics and Stereology.* Eds. R. Ambartzumian & W. Weil Teubner, Leipzig, 110–123.

[10] OREY, S. (1971). *Limit theorems for Markov chain transition probabilities.* Van Nostrand Reinhold, London.

[11] WATSON, G. S.(1986). The shapes of a random sequence of triangles.*Adv. Appl. Prob.* **18,** 156–169.