A mosaic of triangular cells formed with sequential splitting rules

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Abstract

The iterative division of a triangle by chords which join a randomly-selected vertex of a triangle to the opposite side is investigated. Results on the limiting random graph which eventuates are given. Aspects studied are: the order of vertices; the fragmentation of chords; age distributions for elements of the graph; various topological characterisations of the triangles. Different sampling protocols are explored. Extensive use is made of the theory of branching processes.

Keywords: Random topology, random geometry, distribution theory, branching processes, division of space.

AMS 1991 Subject Classification: 60D05, 60E05, 60J80, 68Q80, 68R10.

1. Introduction

Consider a convex polygon with k sides. Suppose there is a well-defined random rule which divides this polygon with a straight chord. Two convex polygons, called the *daughters* of the original *mother cell* result. The rule is now applied independently to each daughter, and to their daughters, and so on. A population of random polygons emerges.

A number of interesting problems can be posed. If geometric entities (distances, angle sizes, areas, etc.) are involved, the problems are random geometric in character, and usually very difficult. I have found, however, a range of interesting stochastic problems which are mainly topological in character.

In Cowan [2] and Cowan and Morris [4], numerous rules involving a chord which hits two randomly chosen sides of the polygon (at any position on the sides but not at a corner of the polygon) were explored, with some fascinating results. For example, in the limit as the number of generations n becomes large, the number of sides S of a randomly chosen polygon converges in distribution. Moreover, if the random chord of a j–gon is selected by an equally-likely choice of the 2 sides to be hit from the $\frac{1}{2}j(j-1)$ pairs of sides, then $S-3$ converges in distribution to the $Poisson(1)$ law.

Other *side-to-side* (SS) division rules in [4] involve avoidance of, or liking for, polygons of x sides. For example, in the so-called *tri–phylic* case when at every opportunity a j–gon divides to give a triangle and a $(j+1)$–gon (by choosing two adjacent sides to be joined by the chord), $S-3$

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converges to a $Geom_0(1)$ distribution, where $Geom_r(\mu)$ denotes Geometric on $\{r, r+1, r+2, \ldots\}$ with mean μ.

I am encouraged to embark on a more systematic study of the *random topology* of polygon division. A key feature of this new work, and indeed that of [2] and [4], is its use of the theory of branching processes. Because Chris Heyde made fundamental contributions to branching-process theory, I hope that this small contribution to his Festschrift is a suitable tribute to his long and productive career.

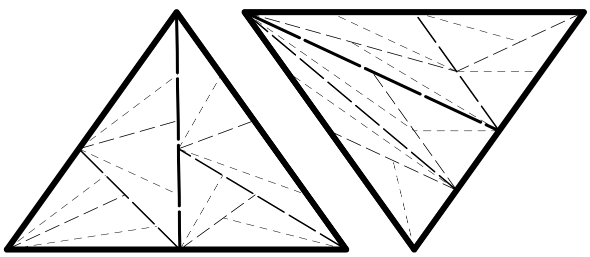
In this paper I shall apply the theory of branching processes to explore questions which arise with *corner-to-side* (CS) rules for the dividing chords. All new chords which divide polygonal cells at successive generations join a corner to a side.

The words *corner* and *side* are used when discussing polygons; I use the words *vertex* and *edge* in the context of the *planar graph* that arises in the iterative division process. Many of my results focus on the randomly-sampled entities of the graph, its vertices, edges and faces (also called *polygons* or *cells*). Other results deal with randomly-sampled elements such as chords, polygon–sides and angles.

Note that the planar graph that is created still incorporates some "geometry" since the chords are straight lines and the faces of the graph are polygons. So there will be, at most vertices in the graph, an angle which is geometrically constrained to be π. The problem of sequentially dividing the faces of a *purely* topological planar graph by *edges* (which are not straight lines) has been studied elsewhere (Cowan and Chen, [3]).

2. Corner–to–side rules

We start with a k–gon at generation 0. Choose a corner of the polygon randomly and then join this corner to a randomly-chosen side (excluding the two sides which are incident at the chosen corner). This straight joining–line, called a *chord*, has a C-end and an S-end, C for *corner* and S for *side*. Iterate the procedure, dividing all polygons in all generations by the same rule applied independently. Eventually, only triangles and quadrilaterals are left. After this happens, the number of quadrilaterals remains static as the generation number n increases, whilst the number of triangles grows like 2^n, dominating the quadrilaterals.



(a) (b)

Figure 1: The corner–to–side division process at generation $n = 4$. In (a), all vertices inside the initial triangle have one angle which is π due to the offset rules mentioned in the text. In the non–offset variant shown in (b), this is not true since the S-end is always placed at the mid–point of the triangle's side (dividing the area into two equal parts).

So let us eliminate the nuisance of quadrilaterals by simply starting with $k = 3$, namely, a triangle (whose boundary we shall denote by Δ). Choose a corner randomly (equally-likely choice) and draw the chord by joining the corner to the opposite side. Then divide the two daughter triangles in the same random way, and so on. Ensure that, when the chord hits the opposite side, the new vertex is a topological T, not a topological X as it could be if a chord from another triangle puts its S-end at the same point. An offset of the two S-ends is employed in this circumstance. Figure 1(a) shows a realisation of the first few generations, whilst Figure 1(b) demonstrates a variant which is presented for contrast (but not analysed in this paper).

3. The basic elements: their prevalence and age profiles.

Let $\mathcal{F}_n, \mathcal{E}_n$ and $\mathcal{V}_n$ be the numbers of faces, edges and vertices of the planar graph at generation n, or as we now prefer to say, time n (each face living for one time unit). For the CS-system with offsets, $\mathcal{F}_n = 2^n, \mathcal{V}_n = 2^n+2$ and, from Euler's planar-graph formula, $\mathcal{E}_n = \mathcal{F}_n+\mathcal{V}_n-1 = 2^{n+1}+1$.

Similarly, we let $\mathcal{S}_n$ and $\mathcal{A}_n$ denote the numbers of sides and angles of extant triangles at time n, and $\mathcal{C}_n$ the number of chords. Clearly $\mathcal{S}_n = \mathcal{A}_n = 3\mathcal{F}_n$ and $\mathcal{C}_n = \mathcal{V}_n$.

Faces, edges, sides and angles are transient elements of our system – they are born and they die – whilst vertices and chords, once created, are permanent, merely changing their character. An *angle* lives for a random time L which follows the $Geom_1(3)$ distribution,

$$\mathbb{P}\{L = \ell\} = \tfrac{1}{3}(\tfrac{2}{3})^{\ell-1} \qquad \ell = 1, 2, 3, \ldots, \tag{1}$$

as does a *side*. Note, however, that lifetimes of angles within the same triangle are not independent; likewise for sides. The lifetime distribution of an edge cannot be stated without further assumptions on the offset rule.

Let A_n be the *age* of a randomly selected vertex at time $n \geq 1$. Clearly there are 2^{n-1-a} vertices of age a, for $a = 0, 1, 2, \ldots, n-1$, *plus* the three original vertices which have age n, so $2^n + 2$ or $\mathcal{V}_n$ in total. So

$$\begin{aligned} \mathbb{P}\{A_n = a\} &= \frac{3}{2^n+2} \qquad a = n \\ &= \frac{2^{n-1-a}}{2^n+2} \qquad a = 0, 1, 2, \ldots, n-1, \end{aligned} \tag{2}$$

which, in the limit for large n, conforms to a $Geom_0(1)$ distribution. The age distribution for extant chords follows the same law, due to the 1–1 correspondence between the new chords and new vertices at each generation.

We shall frequently see, in the course of this paper, equivalent results for chords and vertices; a *chord–vertex duality* exists, as does another duality between angles and sides.

4. Angles and sides: imbedded branching processes.

Each angle, one of the three originals or any newly-created angle later in the process, is the *ancestor* of a simple branching process, one which may be viewed either as a Bellman-Harris (BH) ([5]) or Bienaymé-Galton-Watson (BGW) branching process ([1]).

The Bellman-Harris viewpoint. Suppose the ancestor angle is born at time t. It, and any descendant angle, lives for a random lifetime L distributed as in (1) before 'dying' and producing (immediately) two daughter angles. All lifetimes are independent. This is a classical Bellman-Harris process (also called an age-dependent branching process). One of the standard results for the BH-model is the age distribution of extant individuals at time $t + n$, as $n \to \infty$. Those familiar with this theory may care to show that this limiting age distribution, applied here, is $Geom_0(1)$ – the same as the age distribution in Section 3 for all extant chords and vertices.

We hasten to say that this new result is the age distribution for the extant angles descendant from one angle (and does not describe the ages for all extant angles in the system). The result does, however, reflect a common phenomenon: the age profile of extant individuals in a BH-process has a bias toward young individuals. The mean lifetime of an angle is 3, yet the mean age of extant descendants of a long-dead ancestor angle is 1, less than half of the average life. $\square$

The Bienaymé-Galton-Watson viewpoint. Those less familiar with the BH-process may prefer a more prosaic view. The ancestor angle is again born at time t. It may divide in the next round of division (time $t+1$) into two angles, each of which may then divide into two, and so on. Let Z_n be the number of angles descendant from the ancestor at time $t+n$. Clearly $Z_0 = 1$ and the sequence $Z_0, Z_1, Z_2, \ldots$ forms a Bienaymé-Galton-Watson branching process with 'offspring' pgf given by

$$Q(z) = \tfrac{2}{3}z + \tfrac{1}{3}z^2. \tag{3}$$

The word 'offspring' is to be interpreted as *either* the two daughters if the mother divides *or* one 'daughter' who is actually the surviving mother, reborn. Let H_n be the pgf of Z_n. From the standard theory of BGW branching processes [1], $H_0(z) = z$ and, for $n = 1, 2, \ldots$,

$$H_n(z) = H_{n-1}(Q(z)) = Q(H_{n-1}(z)) = \tfrac{2}{3}H_{n-1}(z) + \tfrac{1}{3}[H_{n-1}(z)]^2 \tag{4}$$

Since the mean of the 'offspring' distribution is $\frac{4}{3}$, $\mathbb{E}Z_n = (\frac{4}{3})^n$. $\square$

Everything in this Section remains true if the word 'angle' is replaced by 'side'. This is an example of the *side-angle duality.*

5. The order of a randomly sampled vertex.

What is the order X_n of a randomly chosen vertex of the random graph at time n? Freshly generated vertices (with the initial T structure) have order 3, but the order of a vertex increases whenever that vertex is chosen as the C-end of a new chord. So, orders of $4, 5, 6, \ldots$ can be created, the population of vertices being constantly fed by fresh vertices of order 3. We shall investigate the distribution of X_n below, but we can find its mean immediately from elementary counting arguments: $\mathbb{E}X_n = 2\mathcal{E}_n/\mathcal{V}_n = (2^{n+2} + 2)/(2^n + 2) \overset{n\to\infty}{\longrightarrow} 4$.

Each *newly created* vertex has two *angles*, ignoring the angle π at the top of the T. Each of these two angles is the ancestor of a BGW-process, governed by (4) and (3). Indeed, there are 2^{n-1} newly created T-vertices at time $n \geq 1$ and the commencement of two BGW-processes of angles at each. Additionally, the three original angles each start one BGW-process. There is some dependence between all these BGW-processes, but it is important to note (when we reach (5) below) that pairwise independence exists for the two BGW-processes which commence at any new vertex.

At time n, when we sample our random vertex, these BGW branching processes are at varying stages of development due to varying ages of the extant vertices (see (2)). Thus the number Y_n of angles (ignoring the one equal to π) at a vertex, randomly–chosen at time n, has pgf D_n, given, for $n \geq 1$, by

$$\begin{aligned} D_n(z) &= \mathbb{E}[\mathbb{E}(z^{Y_n}|A_n)] \\ &= \frac{3}{2^n+2}H_n(z) + \sum_{a=0}^{n-1}\frac{2^{n-1-a}}{2^n+2}[H_a(z)]^2 \qquad (5) \\ &= \frac{3}{2^n+2}H_n(z) + \frac{2^{n-1}}{2^n+2}[H_0(z)]^2 + \sum_{a=1}^{n-1}\frac{2^{n-1-a}}{2^n+2}[H_a(z)]^2 \\ &= \frac{3}{2^n+2}H_{n-1}(Q(z)) + \frac{2^{n-1}}{2^n+2}z^2 + \frac{2^{n-1}+2}{2^n+2}\sum_{b=0}^{n-2}\frac{2^{n-2-b}}{2^{n-1}+2}[H_b(Q(z))]^2 \\ &= \frac{2^{n-1}}{2^n+2}z^2 + \frac{2^{n-1}+2}{2^n+2}D_{n-1}(Q(z)), \qquad (6) \end{aligned}$$

with $D_0(z) = z$. It is a simple matter to show from (6) that $\{D_n(z)\}_{n\geq 0}$ is a Cauchy sequence for $|z| \leq 1$, thereby establishing convergence. This establishes that the sequence $\{Y_n\}$ converges in distribution to a variate, Y say. The limit function, D say, is the solution of the equation

$$D(z) = \tfrac{1}{2}[z^2 + D(Q(z))], \qquad (7)$$

namely

$$D(z) = \sum_{a=0}^{\infty}(\tfrac{1}{2})^{a+1}[H_a(z)]^2. \qquad (8)$$

Whilst I have not found an explicit formula for the coefficient of z^n in $D(z)$, I have

$$D(z) = \tfrac{9}{14}z^2 + \tfrac{27}{161}z^3 + \tfrac{3807}{47012}z^4 + \tfrac{98901}{2667931}z^5 + \cdots \qquad (9)$$

in series form as the generator of Y's probability distribution. Note that $\mathbb{E}Y = 3$ and $\mathbb{V}\mathsf{ar}Y = 12$, provable directly from differentiations of (7) with respect to z. We also note the transient result, $\mathbb{E}Y_n = 3\ 2^n/(2^n + 2)$.

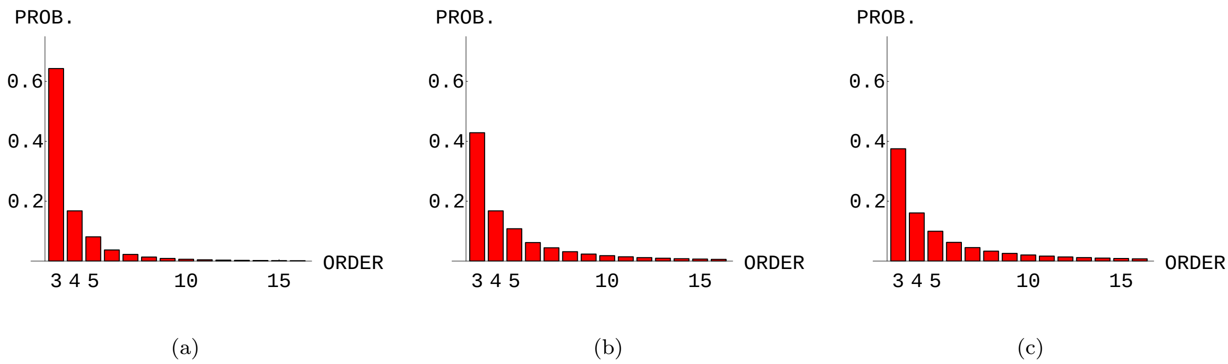


Figure 2: The asymptotic probability mass function of order X for (a) a random vertex, (b) the vertex at a random corner of a random triangle (as per Section 10), (c) a random vertex on the boundary of the graph [as per (21)]

Obviously the order X_n of a random vertex at generation n is $Y_n + 1$; thus $X := Y + 1$ is the variate to which the sequence $\{X_n\}$ converges in distribution. Note $\mathbb{E}X = \mathbb{E}Y + 1 = 4$, as required.

Its distribution, generated by $zD(z)$ and plotted in Figure 2(a), follows from (8) and (9). This is a long–tailed distribution, longer than the Geometric tail in the distribution of age for the randomly selected vertex.

6. Chord fragmentation into edges.

When a chord is first created, it forms a new edge of the planar graph with the C-end of the chord being an existing graph-vertex and the S-end creating a new vertex. Since this new chord has just divided a triangle, it has created two new triangle *sides* on itself. One is a side of the triangle to the chord's left and the other a side of the triangle to the right. Here, we have assigned each chord a direction (running say from its C to S ends).

It is obvious that each of these sides is an ancestor of a BGW-process whose structure is identical to the BGW-angle process discussed above. These two BGW-side processes are independent. Thus, the total number of triangle sides on the chord (both right and left) is akin to the total number of non-π angles at a vertex. The former variate, Y_n^* say, for a randomly sampled chord at time n, is therefore identically distributed to the earlier-defined variate Y_n.

Let I_n be the number of edges on the random chord. Clearly $I_n = Y_n^* - 1 \longrightarrow I$ as $n \to \infty$, I being distributed as $Y - 1$. Note that $\mathbb{E}I = 2$, reflecting the fact that most chords are too young to have achieved much fragmentation. The 'average' chord is therefore one with two sides on the left and one on the right (or *vice versa*) and hence two edges.

7. A branching process of triangles in the C-S system

The trivial branching process is the one related to triangular cells, where each triangle produces two daughters. This has little intrinsic interest *per se*. If, however, we classify the triangles in some way, then we can formulate a multitype branching structure.

The triangular cells that appear in our system can be classified according to the number of their corners which lie on the original boundary, Δ, be it $0, 1, 2$ or 3. If, moreover, there is a sub-classification of the latter two classes according to the number of their sides covering part of Δ, we are able to identify a *multitype branching process* of the triangular cells. It turns out that types {0, 1, 21, 20, 33, 32, 31} may exist at some stage, where *cs* means the *type* of triangle with c corners and s sides $\subset \Delta$.

Each cell produces two daughters, but these may be a mix of types. The mix is determined randomly, but also independently for each cell dependent only on the type of the mother. For example, a mother cell of type 21 will produce two type-21 daughters with probability $\frac{1}{3}$ *or* one type-21 and one type-1 with probability $\frac{2}{3}$ independently of how the other cells are reproducing.

Much of the dynamics of multitype branching processes is understood from the *mean offspring matrix* M, whose (i, j)th element is the mean number of offspring of type j from a mother of type i ([1], [7]). Here, with types listed in the order 0, 1, 21, 20, 33, 32 and 31,

$$\mathbf{M} = \left(\begin{array}{c|c|cc|ccc} 2 & & & & & & \\ \hline \frac{2}{3} & \frac{4}{3} & & & & & \\ \hline 0 & \frac{2}{3} & \frac{4}{3} & 0 & & & \\ 0 & \frac{4}{3} & 0 & \frac{2}{3} & & & \\ \hline & & 0 & 0 & 0 & 2 & 0 \\ & & \frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ & & \frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{2}{3} \end{array}\right). \qquad (10)$$

Entries not shown are zero. We commence with one 33-cell at generation 0. In generation n, let $\mathcal{F}_n^{(j)}$ be the number of j-type triangles ($j = c$ or cs). If the expected values of these 7 variates are summarised in a row vector $\mathbf{m}_n$, we have $\mathbf{m}_0 = (0\ 0\ 0\ 0\ 1\ 0\ 0)$ and $\mathbf{m}_n = \mathbf{m}_0\mathbf{M}^n$. One can show, either by the spectral theory of matrices or by elementary calculations on the sub-matrices of $\mathbf{M}$, that $\mathbf{m}_n$ is (in transpose form)

$$\mathbf{m}_n^T := \begin{pmatrix} \mathbb{E}\mathcal{F}_n^{(0)} \\ \mathbb{E}\mathcal{F}_n^{(1)} \\ \mathbb{E}\mathcal{F}_n^{(21)} \\ \mathbb{E}\mathcal{F}_n^{(20)} \\ \mathbb{E}\mathcal{F}_n^{(33)} \\ \mathbb{E}\mathcal{F}_n^{(32)} \\ \mathbb{E}\mathcal{F}_n^{(31)} \end{pmatrix} = \begin{pmatrix} 2^n + n(n-1)(\frac{2}{3})^{n-1} - 2n(\frac{4}{3})^{n-1} \\ 2(n-2)[(\frac{4}{3})^{n-1} - n(\frac{2}{3})^{n-1}] \\ 4(\frac{4}{3})^{n-1} - 2(n+1)(\frac{2}{3})^{n-1} \\ (n-2)(n-1)(\frac{2}{3})^{n-1} \\ 0 \\ 2(\frac{2}{3})^{n-1} \\ 2(n-1)(\frac{2}{3})^{n-1} \end{pmatrix} \tag{11}$$

for $n \geq 2$. Note, as a check, that the elements of $\mathbf{m}_n^T$ sum to 2^n which equals $\mathcal{F}_n$. Since $\mathcal{F}_n$ is a constant, the probability that a randomly chosen triangle at generation n is of sub–type cs is $\mathbb{E}(\mathcal{F}_n^{(cs)})/\mathcal{F}_n$. In particular, the probability that it is of type 0 can be calculated and shown to approach 1 as $n \to \infty$. Thus, as n gets large, the chance that the cell is of type 0, totally internal to the structure and away from the boundary Δ, tends to 1.

The usual eigenvalue analysis from branching–process theory (see [6]) shows which types become extinct. The square diagonal blocks in $\mathbf{M}$, which represent reproductive rates within the 0, 1 , 2 and 3 cliques, have maximal eigenvalues 2, $\frac{4}{3}$, $\frac{4}{3}$ and $\frac{2}{3}$ respectively. Thus extinction of the 3-clique is certain (because $\frac{2}{3} < 1$). The 2-clique has maximal eigenvalue > 1 yet a breakdown into the two sub–type eigenvalues of this clique confirms the intuitive; $\mathbb{E}\mathcal{F}_n^{(21)}$ grows like $(\frac{4}{3})^n$ and $\mathcal{F}_n^{(20)}$ converges almost surely to zero (since $\frac{2}{3} < 1$) thereby ensuring certain extinction of type 20. The eigenvalues for types 0 and 1 are > 1 and it is easy to show that $\mathbb{E}\mathcal{F}_n^{(1)}$ grows like $(\frac{4}{3})^n$ and $\mathbb{E}\mathcal{F}_n^{(0)}$ like 2^n.

Types 0, 1 and 21 cannot become extinct as each is guaranteed some self-proliferation when it divides, but the latter two types are overwhelmed in number by the faster growing type-0 cells internal to the structure. The theory of [6] allows one to state *almost sure convergence* of the proportionality ratio $\mathcal{F}_n^0/\mathcal{F}_n$ to 1 and of the proportions of the other two surviving types to zero.

One can show that, if $N_E := min_{n>0}\{n : \mathcal{F}_n^{(20)} = \mathcal{F}_n^{(31)} = \mathcal{F}_n^{(32)} = \mathcal{F}_n^{(33)} = 0\}$, namely the first time at which all transient types are extinct, then $\mathbb{E}N_E = 11.33$; 95% of systems have achieved the extinction of transients by time 22 and 98% by time 29.

In generation n, let $\mathcal{V}_n^\Delta$ be the number of vertices of the planar graph on Δ . Simple counting on the graph shows that $\mathcal{V}_n^\Delta$ equals the number of graph–edges on Δ, which equals $\mathcal{F}_n^{(21)} + 2\mathcal{F}_n^{(32)} + 3\mathcal{F}_n^{(33)}$. So, from (11), one sees that

$$\mathbb{E}\mathcal{V}_n^\Delta = 4(\tfrac{4}{3})^{n-1} - 2(n-1)(\tfrac{2}{3})^{n-1}, \tag{12}$$

an entity which grows like $(\frac{4}{3})^n$. Moreover, from the almost-sure convergence results noted above, $\mathcal{V}_n^\Delta/\mathcal{V}_n \longrightarrow 0$ a.s.. So for large n, the chance that a vertex sampled randomly is *internal* is also 1.

8. Other taxonomies for triangles

There are many ways, less primitive than that in Section 7, to define the *type* of an extant triangle.

The $flir$ system: One way is via a classification of each side of a triangle as f, l, i or r. A side which occupies the whole length of a chord (or a whole side of Δ) is labelled f, because the side is a *full* chord. If the chord on which a side lies extends in both directions beyond the side itself, the side is labelled i, because the side is an *internal* part of the chord. If the side lies at the *left* end of the chord when viewed from the triangle, it is labelled l; label r is used for sides at the *right* end of the chord.

Each triangle can then be labelled by a triad of letters. Δ is fff. Its two daughters are ffr and flf, the third generation is some random mix of ffi, ffr, fif, flf, fll, flr, frr, and so on. Note that, for $n \geq 1$, every triangle has at least one f side, namely the side which divided the triangle's mother; we call this the *base side* and write it first in the triad with other labelling done in an anti-clockwise direction.

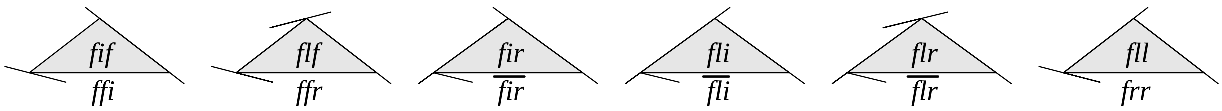


Figure 3: Minimalistic sketches of six $flir$-types which appear in the system. For each, the *base* side is shown at the bottom and directed from left to right (that is, C-end at left, S-end at right). Labelling of each type, shown inside the shaded region, is anti-clockwise starting from the *base*. Other chords (irrelevant to the label and not drawn) may be entrant at the three vertices, but only within their non–shaded angles. Shown below each picture is the anti-clockwise labelling of its reflected image about a vertical line. These 6 reflected types, each with its base directed right-to-left, also appear in the system. To distinguish some cases where the anti-clockwise label remains the same under reflection, an overbar is added.

Our analysis shows that only 13 types can appear in the system; 12 of these survive and are drawn in Figure 3. The other type is fff, the label of Δ. We have analysed this 13-type branching process, but shall not present its $\mathbf{M}$ matrix here. Instead we point out that a classical branching process structure still applies if we coalesce each type drawn in Figure 3 with its reflection. This creates a 7-type process with $\mathbf{M}$ matrix shown below. Coalesced-type labels appear to the right of $\mathbf{M}$.

$$\mathbf{M} = \left(\begin{array}{cccccc|c} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \hline 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{array}\right). \qquad \left(\begin{array}{cc} fif/ffi & \frac{4}{15} \\ flf/ffr & \frac{14}{60} \\ fir/\overline{fir} & \frac{7}{60} \\ fli/\overline{fli} & \frac{7}{60} \\ flr/\overline{flr} & \frac{1}{10} \\ fll/frr & \frac{1}{6} \\ \hline fff & 0 \end{array}\right) \begin{array}{c} 1:1 \\ 1:1 \\ 2:5 \\ 5:2 \\ 1:1 \\ 1:1 \\ \end{array} \tag{13}$$

Here $\mathbf{m}_0 = (0\ 0\ 0\ 0\ 0\ 0\ 1)$. Obviously type fff becomes extinct at time 1. Denote $\mathbf{M}$'s 6×6 sub-matrix of surviving types by $\mathbf{M_s}$. Its maximal eigenvalue, λ say, is 2. One can show, from multitype branching-process theory [6], that the equilibrium proportions of the first 6 types settle to the normalised right eigenvector $\mathbf{p}$ of $\mathbf{M_s}$, namely the normalised solution of $\mathbf{pM_s} = \lambda\mathbf{p}$; the equilibrium vector $\mathbf{p}$ is shown in the display (13).

Also shown is the proportionality within the coalesced types, at equilibrium. I have found these from a full analysis of the 13-type process, because some information is lost from the reduction to 7 types. Note the 5:2 imbalance in some of these proportions.

At time n, there are 3×2^n triangle-sides in the whole graph. One can sample a random one of these by first sampling a random triangle and then sampling one of its sides. A simple calculation, based on the equilibrium of the $flir$–system just derived, shows that, as $n \to \infty$, the chances that the selected side is f, l, i or r are $\frac{1}{2}, \frac{1}{6}, \frac{1}{6}$ and $\frac{1}{6}$, respectively.

Incidentally, if one restricts attention to the pool of 21-cells (see Section 7), one can show that the dominant surviving $flir$-types are fif/ffi, $fli/\overline{fli}$ and $fir/\overline{fir}$. Their equilibrium proportions are the normalised right eigenvector of the matrix $\mathbf{M}_{\mathbf{s}}^{(21)}$, shown below in (14). Here $\lambda = \frac{4}{3} < 2$, reflecting the fact that some daughters of 21-cells are not themselves type-21.

$$\mathbf{M}_{\mathbf{s}}^{(21)} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \qquad \mathbf{M}_{\mathbf{s}}^{(1)} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{14}$$

Clearly, $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ at equilibrium, with further break down to $\frac{1}{6}$ for each of the 6 types (after demerger). On the other hand, the 4 surviving groups of 1-cells are (in order) ffi/fif, flf/ffr, flr and fll/frr, with reproduction rates given by $\mathbf{M}_{\mathbf{s}}^{(1)}$. Their limiting proportions are $\mathbf{p} = (\frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7})$, with proportionalities 1:1 for each merged group.

Thought was given to the merging of labels l and r under one label e, for *extremity*, but this destroys the Markovian structure of the branching process. □

The *ciao* system: Another classification scheme is based on the three angles of the triangle. For nearly all vertices in the graph (save the 3 originals), a key feature is the line which passes *through* the vertex, leaving an angle of π on one side and various smaller angles on the other. Amongst these smaller angles, two are distinct in that the angle is partly created by the *through line*; call these two angles *extremities* and the others, if any, *internal.* Label the extremity angle at the clockwise end of the group c, that at the anti-clockwise end a and all internal angles i. The three original angles of Δ are labelled o (for *original*), but when an o-angle is divided, we apply c, a and i labels in the obvious way.

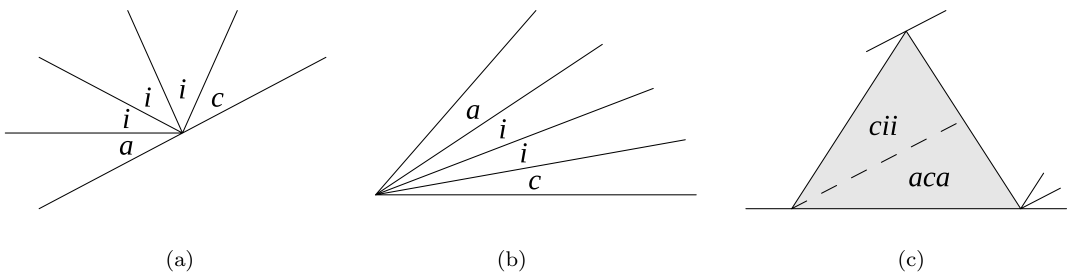


Figure 4: (a) The *ciao*-labelling of angles at a vertex. (b) The labelling at a corner of Δ. (c) The shaded triangle, of *cai*-type, is being divided by the dashed line, realising the $\frac{1}{3}$ chance of *aca* and *cii* daughters.

Each extant triangle now has a triad–labelling inherited from the labels of its angles. The ancestor is *ooo* and, from time 1 onward, we write the triad in an anti-clockwise order starting at the S-end of the base side.

It turns out that 15 triads can appear in the *ciao*–system. Three types, *ooo*, *aco* and *coa* become extinct. The 12 surviving types have labels utilising *three* types of angle (c, i and a), so it comes as a small surprise that there is a 1–1 mapping between these 12 surviving *ciao*-types and the 12 surviving $flir$-types (whose labels involving *four* types of side). The reader can confirm this mapping by observing that the six triangles drawn in Figure 3 are (in order) *cai*, *cii*, *caa*, *cca*, *cia* and *cci*. Their reflections are *aic*, *aii*, *aca*, *acc*, *aci* and *aai*, respectively. Having established the mapping, we can find equilibrium results for the *ciao*-system from our $flir$ results.

The experiment of sampling randomly one of the angles extant at generation n and then randomly one of its angles, yields, in the limit, c, a or i angles with equal probability, $\frac{1}{3}$ each. $\square$

The age of the oldest side (or angle): At time n there are 3×2^n sides of triangles, 3 to each current triangle. The age of a side is defined as the time since that side first became a side of some triangle, either of the current one or one of its ancestors. Note that only one side of a triangle can have an age > 0 and so one can classify triangles by the age of the oldest vertex.

We have a multitype branching process, types being ages $0, 1, 2, \ldots$, starting with a 0-type cell. Population dynamics are governed by

$$\mathbf{M} = \begin{pmatrix} 0 & 2 & & & & & \\ 0 & \frac{4}{3} & \frac{2}{3} & & & & \\ 0 & \frac{4}{3} & 0 & \frac{2}{3} & & & \\ 0 & \frac{4}{3} & 0 & 0 & \frac{2}{3} & & \\ 0 & \frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & \\ \vdots & & & & & & \ddots \end{pmatrix}. \tag{15}$$

It is easily shown from (15) that a, the age of the oldest side of a randomly selected triangle has limiting distribution (as $n \to \infty$) of $Geom_1(\frac{3}{2})$. Since the other two sides have age zero, this implies that the equilibrium age distribution of a randomly sampled side is $Geom_0(\frac{1}{2})$. $\square$

Note that the results for oldest side carry over exactly if the classification scheme is based on the age of the oldest angle (due to the duality between angle and side).

9. The line-of-descent Markov chain

The sampling of a random triangle from the 2^n triangles in generation n can be achieved by sampling in an equally-likely way one of the 2^n lines of descent. At generation 1, one tosses a coin and selects one of the two daughters as the focus of one's attention, repeating the procedure for her two daughters and so on. This is not a recipe that works for all branching processes, merely here, where each mother has precisely two daughters.

When this viewpoint is applicable, the sequence of cell types along the line-of-descent is a Markov chain with transition matrix given by $\mathbf{M}/\lambda$ and the right eigenvector equation becomes the traditional Kolmogorov equilibrium equation for Markov processes. The examples using (13) and (15) can be formulated in this way, but the more *random* multitype branching processes behind (14) cannot. In the latter case, randomly sampling a line of descent is a biased way of sampling a cell of the nth generation.

We consider two further ways of classifying triangles and discuss them from the line-of-descent viewpoint. The first example can also be analysed by the method of Section 8; the second cannot.

The two oldest vertices at the corners (or two oldest bounding chords): Classify a triangle by the ages of the three graph–vertices at the triangle's corners. It is clear that a corner's vertex will be no younger than this corner's angle, and often older. From generation 1 onward, each triangle has only one corner where the graph–vertex has zero age. At the other two corners, the graph–vertices have ages a and b, say, with $a \geq b \geq 1$. So a triangle can be classified as (a, b) according to bivariate age of the oldest two vertices.

Let $p_n(a, b)$ be the probability that the cell type at time n along the line of descent is (a, b). So $p_0(0, 0) = 1$, $p_n(a, a) = 0,\ a \neq n$ and $p_n(n, n) = \frac{2}{3} \times \frac{1}{2} p_{n-1}(n-1, n-1) = (\frac{1}{3})^{n-1}$. Furthermore, for $n \geq 1$,

$$
\begin{aligned}
p_{n+1}(a, b) &= \tfrac{1}{3} p_n(a-1, b-1) && a > b > 1, \\
p_{n+1}(a, 1) &= \tfrac{1}{3}\big(p_n(a-1, a-1) + \textstyle\sum_{b=1}^{a-1} p_n(a-1, b) + \sum_{b \geq a} p_n(b, a-1)\big) && a > 1.
\end{aligned} \tag{16}
$$

Convergence as $n \to \infty$ is easily established, with the consequential dropping of subscript in (16). These Kolmogorov equations can readily be solved to find the limiting bivariate probability mass function for a and b as

$$p(a, b) = (\tfrac{2}{3})^a (\tfrac{1}{2})^b, \qquad a > b \geq 1, \tag{17}$$

with marginals $p_A(a) = (\frac{2}{3})^a [1 - (\frac{1}{2})^{a-1}], a \geq 2$, and $p_B(b) = \frac{2}{3}(\frac{1}{3})^{b-1}, b \geq 1$. Thus, the age of the vertex at a random corner of a random triangle is zero with probability $\frac{1}{3}$, or equal to $x \geq 1$ with probability $\frac{1}{3}[p_A(x) + p_B(x)] = \frac{1}{3}(\frac{2}{3})^x$. So this age $\sim Geom_0(2)$; compare this to the $Geom_0(1)$ law for all extant vertices.

An identical analysis applies if triangles are classified according to the ages of the two oldest bounding chords. □

The Y values of the two oldest vertices: From time 1 onward, there is one angle of every triangle which we call the *base* angle; this is the angle just created by the S-end of the recent dividing chord. The base angle is part of a vertex with $Y = 2$; the other two angles are at vertices with $Y \geq 1$ (and usually ≥ 2 except for small n). Classify each triangle by (Y, y), these two other Y-values; here $Y \geq y$.

When further augmented with the triangle's *ciao*-type, there is a multi-type branching process on triangles, using this classification, but it is not the usual process where each mother reproduces independently of other mothers. There is very complex dependence. Yet, the line-of-descent Markov process is still available for exploitation. Note that the *ciao*-type is important information as it is necessary to know if a triangle's angle is *internal* or not when tracing potential changes to (Y, y).

We leave this fairly lengthy and complicated problem to the reader. The solution, which illustrates how important is the recognition of the line-of-descent Markov chain, is available at www.maths.usyd.edu.au:8000/u/richardc. The same analyis applies to the Y^* values of the two oldest bounding chords. □

10. Y-biased sampling of a vertex.

Consider the sampling of a vertex by first sampling a random triangle and then one of its vertices. Clearly this is a method biased proportionally to the vertex's Y value. So, if $p(y, \mathbf{v})$ is the joint probability mass function of Y and some vector of attributes $\mathbf{v}$ under normal sampling of a random vertex, then the Y-biased pmf is $\frac{1}{3}yp(y, \mathbf{v})$.

As an example of this, choose $\mathbf{v} := a$, the vertex's age. From Section 5 we see that

$$\sum_{y\geq 2} p(y,a)z^y = (\tfrac{1}{2})^{a+1}[H_a(z)]^2.$$

$$\therefore \sum_{y\geq 2} \tfrac{1}{3}yp(y,a)z^{y-1} = \tfrac{2}{3}(\tfrac{1}{2})^{a+1}H_a(z)H_a'(z).$$

$$\therefore \sum_{y\geq 2} \tfrac{1}{3}yp(y,a) = \tfrac{1}{3}(\tfrac{2}{3})^a, \qquad a = 0,1,2,\ldots, \tag{18}$$

in agreement with the earlier result derived from (17). If $\mathbf{v}$ is dropped altogether, we have the simple Y-biased pmf of Y as $\frac{1}{3}yp(y)$, where $p(y)$ is given by (9). Note that $\mathbb{E}Y = 7$. The plot of vertex order, $X := Y + 1$, is seen in Figure 2(b).

11. Following a permanent chord (or vertex).

A chord is permanent. We have established in Section 4, through an exercise with the BH-branching process, the limiting age distribution of extant sides which lie in some chosen chord, τ. Now we ask the nature of a vertex randomly sampled from those lying on τ. Does such a vertex differ in characteristics such as *age* and *order* to the typical random vertex?

The random sampling of a vertex in τ is achieved by sampling a side on τ, randomly from all left and right sides, then taking the vertex at the end of the chosen side nearer to the chord's S-end. This excludes from the sample the C-end of the chord, but this exclusion does not affect the results.

We use the 1–1 correspondence between the vertices and the sides to find the age profile of boundary vertices. Each side can be typed according to the age a of its *linked age*, defined as the age of its linked vertex, $a \in \{0, 1, 2, \ldots\}$. With these types, the population of sides forms another multi-type branching process with *mean offspring* matrix $\mathbf{M}_L$ given below. Also shown is $\mathbf{M}_A$, the matrix which would apply if we used the *actual* age of the side.

$$\mathbf{M}_L = \begin{pmatrix} \frac{1}{3} & 1 & & & \\ \frac{1}{3} & & 1 & & \\ \frac{1}{3} & & & 1 & \\ \frac{1}{3} & & & & 1 \\ \vdots & & & & & \ddots \end{pmatrix}. \quad \mathbf{M}_A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & & & \\ \frac{2}{3} & & \frac{2}{3} & & \\ \frac{2}{3} & & & \frac{2}{3} & \\ \frac{2}{3} & & & & \frac{2}{3} \\ \vdots & & & & & \ddots \end{pmatrix}. \tag{19}$$

It is a simple matter to show from $\mathbf{M}_L$ that, if a side extant in generation n is randomly sampled, its type or *linked age* has limiting pmf (as $n \to \infty$) of $\frac{1}{4}(\frac{3}{4})^a, a = 0, 1, 2, \ldots$.

Turning to the vertices on τ themselves, we have thus shown that their large-n age has the $Geom_0(3)$ distribution. Compare this to the age distribution for all vertices: $Geom_0(1)$.

At each newly-created vertex on τ, the two angles initiate independent BGW-processes having the familiar structure governed by (4) and (3). Thus, by repeating the argument of (5) and (6), one establishes that the vertex order $X^{(\tau)}$ has probabilities generated by $zD_\tau(z)$, where D_τ is the solution of

$$D_\tau(z) = \tfrac{1}{4}z^2 + \tfrac{3}{4}D_\tau(Q(z)). \tag{20}$$

A series solution begins

$$D_\tau(z) = \tfrac{3}{8}z^2 + \tfrac{9}{56}z^3 + \tfrac{513}{5152}z^4 + \tfrac{11745}{188048}z^5 + \cdots . \tag{21}$$

Note that this result also applies to vertices on a given side of Δ and, with slight complication due to the early dependencies between developments on the three sides of Δ, to all vertices on Δ. A plot of the distribution of vertex order $X := Y + 1$ is shown in Figure 2(c). It is easily shown from (20) that the mean order for vertices on τ is infinite.

The analysis above also applies to the Y^* value of a randomly-sampled chord entrant to a chosen vertex. Incidentally, repeating the large-n limit exercise using the matrix $\mathbf{M}_A$ provides an alternative proof to the result given in the "Bellman-Harris viewpoint" of Section 4.

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