Twisted D-modules on affine flag varieties
and Whittaker modules

Anna Romanov
(University of Sydney)

joint with

Emily Cliff
(University of Sydney/Université du Sherbrooke)

The Mathematics of Conformal Field Theory II, Australian National University
Classification of simple Whittaker modules for affine Kac-Moody algebras?
The dawn of geometric representation theory (Beilinson-Bernstein, 1981)

**Beilinson-Bernstein Localisation Theorem**

\[
\text{D}(G/B)_\lambda \text{-mod}_{\mathcal{E}_F} \cong \mathfrak{g} \text{-mod}_{\chi_\lambda} \quad \text{over } \Delta_\lambda
\]

- **semisimple Lie algebra** \(\mathfrak{g}\)
- **G/B flag variety**

\[
\begin{cases}
\mathfrak{g} = \text{sl}(2, \mathbb{C}) \\
G/B = \mathbb{C}P^1 \\
\chi_0 = \text{trivial}
\end{cases}
\]

- Can be used to study essentially all reps of a semisimple Lie algebra. Very powerful!

Dixonter: Any simple \(\mathfrak{g}\)-module of countable dimin has an \(\alpha\) character.

- Ex: finite dim, category \(\mathcal{O}\), \((\mathfrak{g}, k)\)-modules, Whittaker modules, ...

Q: Do we have a BB-type equivalence for (affine) Kac-Moody algebras?

A: not quite...
Why? Essential reason: $\mathfrak{g}$ simple Lie alg/$\mathfrak{g}$ corresp. affine KM alg

- $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{u}(\mathfrak{g})$ is "large" ($\mathrm{HC}$ iso: $\mathfrak{z}(\mathfrak{g}) \simeq \mathfrak{s}(\mathfrak{b})^\mathbb{W}$) $\to$ action tells a lot
- $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{u}(\mathfrak{g})$ is "small" ($\mathfrak{z}(\mathfrak{g}) = \mathfrak{c}[E]$ central extension) $\to$ action only tells level

more precisely: Why can we use $D$-modules on geometric spaces to study reps of Lie algebras?

$G \overset{\text{homogeneous space}}{\overset{\text{differentiable}}{\overset{\text{Lie group}}{\longrightarrow}}} \mathfrak{g} \overset{\text{Lie algebra of } G}{\longrightarrow} \mathfrak{u}(\mathfrak{g}) \overset{\text{morphism of filtered algebras}}{\longrightarrow} \Gamma'(X, D_X)$

$\Rightarrow$ get a functor $\Gamma : D_X$-modules $\rightarrow \mathfrak{g}$-modules

No reason to expect equivalence!

Miracle of BB localization: $G \overset{\text{G/B}}{\longrightarrow}$ (assume $G$ reductive alg/geom)

- $\mathfrak{u}(\mathfrak{g}) \rightarrow \Gamma'(\mathfrak{g}_B, D_{\mathfrak{g}_B})$ factors over $\mathfrak{u}(\mathfrak{g})/(\mathfrak{z}(\mathfrak{g})_+)$ $\overset{\Theta}{\longrightarrow} \Gamma'(G/B, D_{G/B})$

- associated graded:

  $\text{Gr LHS}_\Theta = \mathfrak{S}(\mathfrak{g})/(\mathfrak{S}(\mathfrak{g})_+)$
  $\text{Gr RHS}_\Theta = \Gamma(T^*G/B, \mathcal{O}_{T^*G/B}) = \Gamma(\mathcal{N}, \mathcal{O}_\mathcal{N})$

  Springer nilpotent cone

$\Rightarrow \Theta$ is an isomorphism
So in the special case of the homogeneous space $G/B$, we have equivalences

$$\Gamma : D_{G/B} \otimes_{\text{B-loc}} \sim \Gamma(G/B, D_{G/B}) \otimes_{\text{B-loc}} \sim \mathcal{U}(\mathfrak{g})/\mathfrak{z}(\mathfrak{g})^+ \sim \Gamma(G/B, D_{G/B})$$

Key: $\mathcal{U}(\mathfrak{g})$ is "just the right size" to make $\mathcal{U}(\mathfrak{g})/\mathfrak{z}(\mathfrak{g})^+ \sim \Gamma(G/B, D_{G/B})$.

**THE UPSHOT:** For $\mathfrak{g}$ simple, one can find a homogeneous space $G/B$ s.t. essentially all $\mathfrak{g}$-representations can be studied as $D$-modules on this space.

**Affine case:** $G((t)) \supset G[[t]] \supset G$ \quad Iwahori: $\mathfrak{f} := G((t))/I$ \quad affine flag variety

As before, we have

$$\widehat{G} \supset \mathfrak{f} \quad \sim \quad \widehat{\mathfrak{g}} \quad \rightarrow \quad \Gamma(\mathfrak{f}, D_{\mathfrak{f}}) \quad \rightarrow \quad \mathcal{U}(\mathfrak{g}) \quad \rightarrow \quad \Gamma(\mathfrak{f}, D(\mathfrak{f}))$$

- Factors through $\mathcal{U}(\mathfrak{g})/(C_{\mathfrak{C}}(I)) \rightarrow \Gamma(\mathfrak{f}, D(\mathfrak{f}))$

**THE UPSHOT:** Can't expect to be able to study all $\mathfrak{g}$-representations using $D(\mathfrak{f})$-modules.

---

\(\text{No reason to expect isomorphism! (No Kostant normality theorem)}\)
So should we give up hope of studying $\mathfrak{g}_0$-reps using D-modules?

No! Lots of functors $\Gamma : D\text{-modules on space }X \to \hat{\mathfrak{g}}\text{-modules},$ we just don't expect equivalences

- Some useful spaces:
  - Affine Grassmannian $Gr := G(\mathfrak{l}) / G(\mathbb{C} \mathfrak{l})$
  - Affine flag variety $Fl := G(\mathfrak{l}) \backslash G(\mathfrak{t})$
  - Enthanced affine flag variety $\hat{Fl} := G(\mathfrak{l}) \backslash I_u$ equi($\mathfrak{n}$) where $\mathfrak{n} \subset \mathfrak{b}$ unipotent radical

A natural question:

Q: Which $\hat{\mathfrak{g}}_0$-reps can we obtain from D-modules on a given space?

- Example: (Kashiwara-Tanisaki '97, see also Frenkel-Gaitsgory '04) K negative level (i.e. $k + h' \in \mathfrak{a}_+$)

$$\Gamma : D(\hat{Fl})_k \text{-mod} \xrightarrow{\text{equiv}} (\mathcal{O}_{\text{aff}})^{\lambda}_k$$

$\hat{\mathfrak{g}}_0$-modules in negative level blocks of category $O$ are global sections of D-modules on $\hat{Fl}$

$\sim$ at the negative level, $\mathcal{O}_{\text{aff}}$ is in essential image of $\Gamma : D(\hat{Fl})_k \text{-mod} \to \hat{\mathfrak{g}}_0 \text{-mod}$
So modules in $\widehat{O}$ can be studied as $\widehat{g}$-modules on (enhanced) affine flag varieties. (We can twist differential operators to work with modules over TDOs on $\widehat{F}_0$ instead of $\widehat{g}$-modules on $\widehat{F}_0$)

Q: Which $\widehat{g}$-modules are not in the essential image of $\pi^* : D(\widehat{F}_0) - \text{mod} \to \widehat{g} - \text{mod}?$

Emily and I think we found some nondegenerate Whittaker modules

First: 1. What are Whittaker modules? 2. What are $\widehat{g}$-modules on ind-schemes?

1. $\widehat{g} = g \otimes_{\mathbb{Z}} \mathfrak{h} \otimes \mathfrak{n}_+ \xrightarrow{\eta} \mathfrak{n}_+$ Lie algebra morphism

   def: An $\eta$-Whittaker module is a $\widehat{g}$-module $V = U(\mathfrak{g}) w$ with $X \cdot w = \eta(x) w \ \forall \ X \in \mathfrak{n}_+$.

   def: $\eta : \mathfrak{n}_+ \to \mathbb{C}$ is nondegenerate if $\eta|_{\mathfrak{l}_a} \neq 0 \ \forall \ a \text{ simple}$

A category: \((\widehat{g}, I_u, \eta) - \text{mod}\)

objects: \((V, \pi, \mu) \text{ with } \pi : \widehat{g} \to \text{End}(V), \mu : I_u \to \text{Aut}(V)\)

s.t. (i) $\widehat{g} \otimes V \to V$ action map is $I_u$-equivariant

(ii) $\Delta \mu(x) - \pi(x) = \eta(x) \ \forall \ x \in \mathfrak{n}_+$

morphisms: linear maps compatible w both structures

A home for Whittaker modules!
What are $D$-modules on ind-schemes?

**Approach 1 (Kashiwara–Tanisaki):**

- Define $\mathcal{F}_\ell = \bigsqcup_{w \in W_{\text{eff}}} X_w$, with $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = \bigsqcup_{\ell \in \mathbb{N}} X_{\ell n}$.
- This gives $\mathcal{F}_\ell$ the structure of a projective ind-variety.
- $\mathcal{F}_\ell = \bigsqcup_{n \geq 0} X_n$.
- $D(\mathcal{F}_\ell)$-modules $= \varprojlim_{n \to \infty} D(X_n)$-modules.
- Works for any ind-variety.
- Can add twists for $\lambda \in \mathbb{A}^1$ (include level information) $\rightsquigarrow D(\mathcal{F}_\ell)_{\lambda}$-mod.

**Approach 2 (Arkhipov–Gaitsgory):**

- Not an ind-variety!
- Develop theory of $D$-modules on $G(\mathfrak{gl}(t))$ (ind-pro-scheme of finite type).
- Define $D(G(\mathfrak{gl}(t)/K)$-modules $: = (D(G(\mathfrak{gl}(t))$-modules)$)^K$.
- Define $D$-modules on $G(\mathfrak{gl}(t))$: modules over chiral algebra $\mathcal{O}_{G, K} := \text{Ind}_{G(\mathfrak{gl}(t)/K)}^{G}$.
- Claim: This construction is equivalent to the first when $G(\mathfrak{gl}(t)/K$ is an ind-variety.
- The point of this is that it's useful (essential?) at the critical level.

**KEY:** In either formulation, $D$-module functor formalism holds.
Our construction: Fix $k, G, \Lambda$ level

- $\eta: I_u \to G_a$ additive character $\to \mathcal{L}_\eta = \eta^* \mathbf{e}^x$ character sheaf on $I_u$
- $I_u \times \mathfrak{g} \to \mathfrak{g} \lambda \lhd \mathfrak{g}$ action map

Definition: A $\eta$-twisted $I_u$-equivariant $D(\mathfrak{g})_k$-module is a pair $(U, \Phi)$ with $U \in D(\mathfrak{g})_k$-mod and $\Phi: \text{ad}^* U \to \mathcal{L}_\eta \otimes U$ iso of $D(I_u \times \mathfrak{g})_k$-modules.

Form a category $D(\mathfrak{g})_k$-mod\text{I}_u, \eta

- Standard objects: $\xymatrix{ \xi: \mathbf{w} \otimes \mathcal{L}_\eta \ar[d] \ar[r] & I_u \otimes \mathcal{L}_\eta \ar[d] \ar[r] & I_u \ar[d] & I_u \otimes \mathcal{L}_\eta \ar[d] \ar[r] & I_u \otimes \mathcal{L}_\eta }$

- Simple objects: $\xymatrix{ \xi: \mathbf{w} \otimes \mathcal{L}_\eta \ar[d] \ar[r] & I_u \otimes \mathcal{L}_\eta \ar[d] \ar[r] & I_u \ar[d] & I_u \otimes \mathcal{L}_\eta \ar[d] \ar[r] & I_u \otimes \mathcal{L}_\eta }$

Standard objects $\otimes \mathcal{L}_\eta = \eta$-twisted structure sheaf on Bruhat cell $\mathbf{w}$ (only exists on certain Bruhat cells).

How to parametrise simple objects:

\[
\begin{align*}
\left\{ \text{Simple objects in } D(\mathfrak{g})_k\text{-mod}_{I_u, \eta} \right\} & \overset{1:1}{\longrightarrow} \left\{ \text{Bruhat cells admitting } \mathcal{O}_w \right\} & \overset{1:1}{\longrightarrow} \left\{ \text{In-orbits $X_w$ s.t. } \eta \big| \text{Stab}_{I_u} \mathbf{x}_w = 1 \times \mathbf{x}_w \right\}
\end{align*}
\]
By construction,

\[ \Gamma: D(\mathfrak{sl}_k)_{k-\text{mod}} \rightarrow (\hat{\mathfrak{g}}_k, \mathbb{I}_w, \eta)_{\text{mod}} \rightarrow \eta-\text{Whittaker modules of level } k \]

A tool for studying Whittaker modules?

Our category from earlier!

Utility depends on level:

- \( k \) negative
  \[ \Gamma \text{ exact + fully faithful} \]
  (Beilinson-Drinfeld)

- \( k \) critical or positive
  \[ \Gamma \text{ not exact, not full or faithful} \]
  (Frankel-Gaitsgory)

Assume \( k \) negative.

**Conjecture:** The only \( D(\mathfrak{sl}_k)_{k-\text{mod}} \) modules whose \( \Gamma \) are in \( (\hat{\mathfrak{g}}_k, \mathbb{I}_w, \eta)_{\text{mod}} \) are in \( D(\mathfrak{sl}_k)_{k-\text{mod}} \).

The nondegenerate case:

**Lemma:** For any \( x \in \mathfrak{sl}_k \), \( \text{Lie} (\text{stab}_{\mathbb{I}_w} x) \) contains a simple root space.

**Recall:**

\[ \begin{cases} \text{Simple objects} \rightarrow \text{I}_w-\text{orbits } X_w \rightarrow \text{I}_w-\text{orbits } X_w \text{ with } \eta |_{\text{stab}_{\mathbb{I}_w} x} = 1 \end{cases} \]

**Lemma \Rightarrow:** If \( \eta \) is nondegenerate,

\[ \eta |_{\text{stab}_{\mathbb{I}_w} x} \neq 1 \text{ for any } x \in \mathfrak{sl}_k \]

**Consequence:** \( D(\mathfrak{sl}_k)_{k-\text{mod}} \mathbb{I}_w \eta \) is empty if \( \eta \) is nondegenerate!

Hence, if conjecture is true, nondegenerate Whittaker modules are NOT in the essential image of

\[ \Gamma: D(\mathfrak{sl}_k)_{k-\text{mod}} \rightarrow \hat{\mathfrak{g}}_k-\text{mod} \]
THE UPSHOT: We still don’t know what the essential image is, but now we know some modules which are in it (category $\mathcal{O}$) and some modules which aren’t (nondegenerate Whittaker modules).

A bit more:

- We suspect $\Gamma: D(\mathcal{J}_L)_k \text{-mod} \xrightarrow{I_{\mu, \gamma}} (\hat{g}_k, I_{\mu, \gamma}) \text{-mod}$ is an equivalence for partially degenerate $\gamma$ (if $k$ negative).
- Could nondeg. Whittakers appear as $\Gamma$ of $D$-modules on another space?
- What about positive + critical level?

Thanks for listening!