Question: What are the multiplicities of $L(w\lambda)$ in $M(v\lambda)$ for $\lambda \in \mathfrak{g}^*$?

Gufang: $\lambda = -\rho$ (+p? I was confused by his notes...)

\[ \begin{align*}
\{ & \begin{array}{c}
\begin{array}{c}
\text{BB} \\
\text{BB}^+ \\
\text{BB}^* \\
\text{BB}^+ \\
\text{BB}^*
\end{array}
\end{array} \\
\begin{array}{c}
\mathcal{O}_{-\rho} \longrightarrow \\
\mathcal{D}_B \text{-mod} \longrightarrow \\
\mathcal{D}_B^\text{mod} \longrightarrow \\
\mathcal{D}_B^\text{shif-fact} \longrightarrow \\
\mathcal{H}
\end{array}
\end{align*} \]

\[ \begin{align*}
\mathcal{M}_w & \quad \text{i} \omega! \mathcal{O}_{B_w} \\
\mathcal{L}_w & \quad \text{i} \omega!* \mathcal{O}_{B_w}
\end{align*} \]

Answer: $\lambda = -\rho$

\[ [\mathcal{M}_w, \mathcal{L}_v] = \mathcal{P}_{w, w, w}(1) \]

Kazhdan–Lusztig polynomials

Gufang's convention: $M(v\lambda) := U(g) \otimes \mathbb{C}^\vee \otimes \mathbb{C}^\vee$

(Also Dragon's convention)

Okay, this answers our question for one value of $\lambda$. What about everything else?

integral $\lambda$: tensoring with line bundles immediately gives the answer

In $\mathcal{D}$-module categories have twist functor: For $\lambda \in P(\mathfrak{g})$

\[ (\lambda): \mathcal{D}_B \text{-mod} \longrightarrow \mathcal{D}_{\lambda} \text{-mod} \]

- Equivalence of categories
- Sends standards to standards
- Simplifies to simples

Answer: $\lambda \in P(\mathfrak{g})$

\[ [M(w\lambda) : L(v\lambda)] = \mathcal{P}_{w, w, w}(1) \]

- What about general $\lambda$? 

Strategy: a rational

\[ \begin{align*}
\mathcal{O}_\lambda & \quad \text{BB-source} \\
\begin{array}{c}
\text{BB-source} \\
\text{BB-source}^+ \\
\text{BB-source}^* \\
\text{BB-source}^+ \\
\text{BB-source}^*
\end{array} \\
\begin{array}{c}
\mathcal{O}_{-\rho} \longrightarrow \\
\mathcal{D}_B^\text{-modules} \longrightarrow \\
\mathcal{D}_B^\text{modules} \longrightarrow \\
\mathcal{D}_B^\text{shif-fact} \longrightarrow \\
\mathcal{K}_C \mathcal{L}w_n
\end{array}
\end{align*} \]

Confession: It's not very clear to me how Lusztig's construction actually answers the multiplicity question, and his paper doesn't explain it. The strategy above is how I think it works very roughly, but perhaps after I explain the construction you all can help me understand better the translation to $\mathcal{O}_\lambda$.

Today's goal:

- Sketch of Lusztig's construction
- An alternate approach

Plan:
Lusztig's Approach:

Warning: I'm very sketchy on this machine, so I can only really sketch this construction... the details are a bit out of my reach.

Set-up:
- 1st, a general construction: \( H \times X \) all over \( \mathbb{F}_q \), \( q = p^e \), fix \( l \neq p \) prime.

Choose \( \mathbb{F}_q \)-rational structures on \( H \) and \( X \). Frobenius \( F: X \to X \).

Recall Dougal:

\[
\begin{align*}
X = X_0 \otimes_{\mathbb{F}_q} k, & \quad X_0 = \text{\( \mathbb{F}_q \)}-\text{variety} \\
\text{\( \mathbb{F}_q \)}-\text{rational structure} \\
\text{ex}: X = \text{spec} A, & \quad A = A_0 \otimes_{\mathbb{F}_q} k \\
\text{geometric: } a \otimes \lambda \mapsto a^e \otimes \lambda & \quad \text{raises coordinates to } q^{th} \text{ power} \\
\text{arithmetic: } a \otimes \lambda \mapsto a \otimes \lambda^e & \quad \text{raises coefficients to } q^{th} \text{ power}
\end{align*}
\]

- We will use this one: geometric Frobenius \( F_0: X_0 \otimes k \to X_0 \otimes k \).
- \( F_0 \in \text{End}(X_0) \) raises facts on \( X_0 \) to \( q^{th} \) power.

Build a category \( C_X \):

- objects: \((F, \phi)\), \( \phi \) sheaves + sauce constructible, \( H \)-equivariant, sheaf on \( X \)

- morphisms: \( H \)-equivariant maps commuting with all \( \phi_m \), isomorphisms of sheaves \( \Phi = \{ \phi_m \in (F^m) \to F^m \mid m \geq 1 \} \).

- Eigenvalues of \( \phi \): Sauce = eigenvalues of Frobenius \( \phi \) gives us "mixed" structure.

- For \( x \in X \), have \( \phi_m: F_x \to F_x \) invertible linear transformation.

- Define eigenvalues of \( \phi \) on \( F_x \) to be \( m^{th} \) roots of eigenvalues of \( \phi_m \) elements of \( \mathbb{Q}_l \).
Next: Specialize this construction

\[ T \subset B \subset G \quad \text{connected reductive group, } k = \mathbb{F}_p \]

For \( \ell \in \mathcal{L} \), denote \( L^* = L \) w/o zero section, \( \pi: L^* \to \mathcal{B} \)

Apply general construction to \( H = U \times k^\times \)

\[ \text{action:} \quad \cdot \text{Fix } n \geq 1, (n,p) = 1 \]
\[ \cdot (u, x) \cdot (g, z) = (u \cdot g, x^n z) \]

\[ \text{category } \mathcal{C}_{L^*, n} \]

Goal: Find some bases for the Grothendieck group \( K \mathcal{E}_{L^*, n} \)

- orbits: \( L^* = \bigsqcup \pi^{-1}(B_w) \) \quad Brion cells \( B = \bigsqcup B_w \)

- isotropy groups: \( \text{Stab}_H \ni u \times M^n \subset U \times k^\times \)

For \( n \neq 1 \), isotropy groups are disconnected!

Some sheaves:

\[ \psi: M^n \to k^\times \]

\[ \mathcal{E}_{w, \psi}^L \text{ unique } H\text{-equiv locally constant } \underline{k}\text{-sheaf of rank } 1 \text{ over } \pi^*(B_w) \subset L^* \]

Extends by zero to \( L^* \), call by same name

We're trying to find a basis for \( K \mathcal{E}_{L^*, n} \). Objects in \( \mathcal{C}_{L^*, n} \) are sheaves + a space, so to turn \( \mathcal{E}_{w, \psi}^L \)'s into a basis we need to put a \( \phi \)-structure on them.
For $a \in \mathbb{A}$, $\pi^! \Phi$-structure on $X_{w,\psi}^L$ s.t. $\Phi$ has eigenvalue $a$.

* Parsing this:
  - $\Phi_\psi$ is $H$-equivariant $\pi^!$-action on $\overline{Y}_w$.
  - $X_{w,\psi}$ is rank 1, so on stalks, $\Phi_\psi$ is just multiplication by a $\#_w$ in $\mathbb{C}_x$.
  - So if we fix $a \in \mathbb{A}$, there is a unique collection of $\Phi_\psi$'s s.t. this $\#_w$ is $a^w$.

We get a collection of elements in $KE_{L,n}$:

$$a\overline{X}_{w,\psi}^L := \left[ \left( X_{w,\psi}^L, \Phi(a) \right) \right] \in KE_{L,n}$$

- these form a $\mathbb{Z}$-basis of $KE_{L,n}$.
- Give $KE_{L,n}$ structure of a $\mathbb{Z}[A]$-module by
  $$a \cdot \overline{X}_{w,\psi}^L = a\overline{X}_{w,\psi}^L$$

**The upshot:** Our first basis of $KE_{L,n}$ (as a $\mathbb{Z}[A]$-module) is $\left\{ \overline{X}_{w,\psi}^L \right\}$.

Next we construct a second basis.

* Tool: Deligne, Goresky-MacPherson intersection cohomology

Input: $X_{w,\psi}^L$,

$$\xrightarrow{\text{Sheaf on open dense subset}} \mathfrak{IC} \left( \pi^!(\overline{B}_w), X_{w,\psi}^L \right) \in D^b \left( \pi^!(\overline{B}_w) \right)$$

- complex in derived category s.t. cohomologies
- $H^i \left( \pi^!(\overline{B}_w), X_{w,\psi}^L \right)$ constructible $H$-equivariant $\mathbb{A}_c$-sheaves on $\pi^!(\overline{B}_w)$

- extend trivially to $L$,
  $$H^i \left( \pi^!(\overline{B}_w), X_{w,\psi}^L \right) \in \mathcal{C}_{L,n}$$

- constructible $H$-equivariant $\mathbb{A}_c$-sheaves on $L$,
- $\Phi$-structure inherited from $X_{w,\psi}^L$.

Define a second basis of $KE_{L,n}$:

$$\left\{ \overline{\xi} := \sum_i (-1)^i \overline{H}^i \left( X_{w,\psi}^L \right) \right\}$$

**class in $KE_{L,n}$ of object**

We've constructed two bases of a Grothendieck group as a $\mathbb{Z}[A]$-module. You can probably guess where this is going. Our next step is to write one basis in terms of the other and analyze the coefficients.
Lemma 1: \( \tilde{\mathcal{X}}_{w,\psi}^L \) is uniquely characterized by two properties:

1. \( D(\tilde{\mathcal{X}}_{w,\psi}^L) = [p]^{(\ell(w) - 1)} \tilde{\mathcal{X}}_{w,\psi}^L \) \( \text{here } [p] \in A \text{ is the image of } p \text{ in } \mathcal{A} \)

2. \( \tilde{\mathcal{X}}_{w,\psi}^L = \tilde{\mathcal{X}}_{w,\psi}^L + \mathbb{Z} [A]\text{-linear combo of } \tilde{\mathcal{X}}_{w,\psi}^L \text{ for } w < w' \)

where coefficients are \( \mathbb{Z} \)-linear combos of \( a \in A \) that are represented by algebraic 1s in \( \mathcal{A} \), whose complex conjugates all have absolute value \( \leq p^{\varepsilon \mathcal{L}(w) - \varepsilon(w - 1)} \).

- Here \( D: KE_{L,n} \rightarrow KE_{L,n} \) is Verdier duality.
  - Some properties:
    - \( D \) is \( \mathbb{Z} \)-linear.
    - \( D(a \cdot x) = a \cdot D(x) \) for \( a \in A \).
    - \( D(\tilde{\mathcal{X}}_{w,\psi}^L) = \sum_{w < w'} r_{w',w,\psi} \tilde{\mathcal{X}}_{w',w',\psi}^L \) where \( r_{w',w,\psi} \in \mathbb{Z} [A] \).

Pause. Examine case \( n = 1 \):

- \( L^\times = \xi(g,z) \in G \times k^\times \) where \( b \cdot (g,z) = (gb^b, x(b)z) \).

- Action \( H \bowtie L^\times \) is \( (u,x) \cdot (g,z) = (u \cdot g, xz) \).

- Isotropy groups \( \text{Stab}_H L^\times = U_1 \subset U \times k^\times \) connected \( \rightarrow \) only one local system on each orbit.

- Over Bruhat cells, have trivial \( k^\times \)-bundle \( \Pi^{-1} \bigl( B_w \bigr) \cong U/U_1 \times k^\times \) true for any \( n \).

\[ \Rightarrow H \text{-equivariant sheaves on } \Pi^{-1} (B_w) \cong U \text{-equivariant sheaves on } B_w \]

\[ \xymatrix{ \Pi^{-1} (B_w) \ar[r] & U \} \}

\[ \text{acts freely on fibres only true for } n = 1 \ldots \]

\[ B_w \cong U/U_1 \]

\[ \Rightarrow \text{can work w/ } U \text{-equivariant } \overline{\mathcal{F}} \text{-sheaves on } B, KE_{L,n} \text{ can be identified w/ Hecke algebra } H, \]

\[ \text{bases } \overline{\mathcal{X}}_{w}^{L} \leftrightarrow q^{\ell(w)/2} T_w \]

\[ \tilde{\mathcal{X}}_{w}^{L} \leftrightarrow C_w \]

\[ \text{using Gutfand's conventions...} \]

The upshot: This construction collapses to the setting of \( U \)-equivariant \( \overline{\mathcal{F}} \)-sheaves on \( B \) when \( n = 1 \).
Why is lemma true?

- $\overline{X}_{w,v}$ satisfy (1.a) and (1.b) by def of $IC(\cdot)$ and the "purity theorem of Gabber." I don't know what this theorem says. They cite BBD, but I don't know where in BBD to find it. I also don't speak French.

Goodie's interpretation: if $\mathcal{F}$ is a pure perverse sheaf on an open subvariety $j: U \rightarrow X$ (see footnote) then so is $j_*\mathcal{F}$. 

$IC(\pi^-_*(\mathcal{B}_x), \overline{X}_{w,v})$ is pure, so Frobenius acts in a very special way on its cohomologies.

This must imply the condition on the coefficients of $\overline{X}_{w,v}$ in Lemma

- Uniqueness straightforward and combinatorial.

**Lemma leads to**  **Question**: What are the coefficients of $\overline{X}_{w,n}$ in (1.b)?

To answer, we need $n$

**Third basis of $K E L \cdot n$**:

- $W$ acts on $L = \mathcal{E}_G$-equiv line bundles over $\mathcal{B}_x / \nu$

  - G-orbits on $B \times B$ are $\mathcal{O}(w) = \mathcal{E}(B', B'') \in B \times B \mid B' \text{ and } B''$ in rel pos $w$

- Let $L_w = \left\{ \text{G-equiv line bundles on } \mathcal{O}(w) \text{ w compatible} \right\}$

- Two projections

  \[
  \begin{array}{ccc}
  B & \xrightarrow{pr_1} & L_w \\
  B & \xrightarrow{pr_2} & L_w \\
 \end{array}
  \]

- $\mathcal{O}(w)$

- $pr_1^*: L \rightarrow L_w$ bijective

- $pr_2^*: L \rightarrow L_w$

- Given $L \in \mathcal{E}_G$, $\exists ! \: \nu \in \mathcal{L}$ s.t. $\exists$ $G$-equiv iso (our $\mathbb{F}_p$) $pr_1^*(\nu(L)) \sim pr_2^*(L)$

- Defines action of $W$: $w \cdot L = \nu(L)$

  (Note: under the association $L \rightarrow X(T)$ this aligns w/ the natural action of $W$ on $X(T)$)

  - Given $L, r, n$, define subgroup of $W$:

    - $W_{L,n} = \left\{ \mathcal{E} \in W \mid w \cdot L = L \otimes L^{\otimes n} \text{ for some } L, \nu \in \mathcal{L}_{\text{root}} \right\}$

    - Coxeter group

    - $W_{L,n} = W$

    - $\tilde{R}^{-n} = \tilde{R}^n$

    - $R_{L,n}^r = \left\{ \mathcal{E} \in R^r \mid \mathcal{E}(L) \text{ is divisible by } n \right\}$

    - $\sim \: R_{L,n}, R_{L,n}^r, R_{L,n}^-, S_{L,n}$
Some properties of $\leq_{L,n}$:
- Any coset $wW_{L,n} \in W/W_{L,n}$ has unique min length elt $w, eW$
- If $v' \leq_{L,n} v$ and $w, eW_{L,n}$ is min length, then $w, v' \leq_{L,n} w, v$ and $\ell(w, v') - \ell(w, v) \geq \ell(v') - \ell(v)$

With this we can define our third basis.

- Given $L, n, \psi: M_n \to \overline{O}_L$, define

$$\hat{\mathcal{X}}^L_{w, \psi} = \sum_{v' \leq_{L,n} v} \left[ \frac{1}{2} \left( \ell(w, v) - \ell(w, v') - \ell(v) - \ell(v') \right) \right] \mathcal{P}_{v, v'}(\psi) \mathcal{X}^L_{w, v'}$$

Here $\mathcal{P}_{v, v'}(\psi)$ is min length in $wW_{L,n}$ and $w = wv$.

KL polys for $(W_{L,n}, S_{L,n})$

This establishes that the coefficients in Lemma 2 are given by KL polys for the integral Weyl group $W_{L,n}$.

Main Lemma: $\hat{\mathcal{X}}^L_{w, \psi} = \hat{\mathcal{X}}^L_{w, \psi}$

geometric basis combinatorial basis

Strategy:
- Show $\hat{\mathcal{X}}^L_{w, \psi}$ satisfies (1.a) and (1.b)

(1.b) follows from properties of KL polys

The work is in showing (1.a) needs to establish

$$D(\hat{\mathcal{X}}^L_{w, \psi}) = [\psi]^{-\ell(w) - 1} \hat{\mathcal{X}}^L_{w, \psi}$$

Approach:
- Define push-pull operators $T_5, T_5': K\mathcal{C}^{L,n} \to K\mathcal{C}^{L,n}$

Recall $W$-action gives $H$-equiv. iso

$$\pi_5 \circ pr_5^{*}(sL) \xrightarrow{k} pr_5^{*}L$$

$$\pi_5 \circ pr_5^{*}L \xrightarrow{\pi_5} pr_5^{*}L$$

$$T_5(S) = \bigoplus (-1)^{r} R^{i} \pi^{*}_{L} (\psi^{*} \pi^{*}_{L} S)$$

$$T_5'(S) = \bigoplus (-1)^{r} R^{i} \pi^{*}_{L} (\psi^{*} \pi^{*}_{L} S)$$

Describe how they interact with $D$, $\hat{\mathcal{X}}^L_{w, \psi}$ and $\overline{\mathcal{X}}^L_{w, \psi}$

- $DT_5 = [\psi]^{-1} T_5'D$
- $T_5(\overline{\mathcal{X}}^L_{w, \psi}) = T_5'(\overline{\mathcal{X}}^L_{w, \psi}) = \begin{cases} 
\overline{\mathcal{X}}^L_{ws, \psi} & ws > w \\
[\psi] \overline{\mathcal{X}}^L_{ws, \psi} & ws < w 
\end{cases}$
- $T_s(\hat{X}_{w_s, w}) = \hat{X}_{w, w}$ if $w_s < w$

- prove by induction in $\ell(w)$ using properties of $T_s$ established above

- need to treat two cases $s \in W_{L \alpha}$ and $s \notin W_{L \alpha}$ separately

This establishes the main result of Lusztig by giving a topological interpretation for the multiplicities in the Jordan-Hölder series of a Verma module of rational highest weight.

- Two big gaps in my understanding:
  1. Exactly how we pass this construction through RH and BB localization
  2. How we move from $k = \overline{F}_p$ to $C$ (see De IF a $C$?)]

This machine of $\ell$-adic sheaves is not my cup of tea. All of this should be able to be done using the machinery of mixed Hodge modules, but that's a world I am not yet competent in. However, to answer our original question, we actually don't need either $\ell$-adic sheaves or MHM. In my last minutes of the talk, I'll sketch an argument using only $D_\alpha$-modules. It's only a sketch because I haven't completely filled in the details yet. But if you ask again in a few months, hopefully Dragon and I will have written the whole thing down carefully...

A $D$-module approach:

- BB localization works for any $\lambda \in \mathfrak{g}^*$: $U_{\theta} \text{-mod} \xrightarrow{\Delta_{\lambda}} D_\lambda \text{-mod} \quad \Theta = W - \lambda$

\[\Rightarrow\] Computing

\[\left[ I(\omega, \lambda) : I(\nu, \lambda) \right]\]

answers our original question.

- Define a map $D_{\alpha} \text{-mod} \xrightarrow{\gamma} \text{free } \mathbb{Z}[q, q^{-1}] \text{-module}$

\[F \mapsto \sum_{w \in W} \sum_{m \in \mathbb{Z}} \dim_q (R^m i_{\gamma^1}(F)) q^m S_w\]

- Apply $\gamma$ to $I(\omega, \lambda)$, get polynomials

\[R_{w \nu} = \sum_{m \in \mathbb{Z}^+} \dim_q (R^m i_{\gamma^1}(I(\omega, \lambda))) q^m\]

- Evaluation $\nu(-1)$ factors through Grothendieck group $[I(\omega, \lambda)] = [I(\omega, \lambda)] + \sum_{\nu < \omega} R_{w \nu} \nu[I(\omega, \lambda)]$
So what are these polynomials?

- If $\lambda = -\rho$, can show $R_{uv}$ satisfy defining relations of Kazhdan–Lusztig polynomials i.e. $D^\lambda_x = D_x$, no twist → see Dragon's Localization notes

*Key tool: Decomposition Theorem

- For arbitrary $\lambda$, we still have decomposition theorem (locally semisimple $\Leftrightarrow$ semisimple) (for holonomic $\mathcal{D}$-modules)

  - Claim: $R_{uv}$ are KL polys for $(W_\lambda, S_\lambda)$

    integral Weyl graph, gen'd by $S_\lambda$ for $\sigma \in R$ which pair integrally $uv / \lambda$

    steps: • Let $v, w \in W_\lambda$. If $R^m_{v^1}(I(w, \lambda)) \neq 0$ for some $m \in \mathbb{Z}$, then $v < w W_\lambda$

      only nonzero $R_{uv}$'s come from $wW_\lambda$ in same right $W_\lambda$-coset

      so the polys are at least the "right size"

    • proof by induction in $I^\lambda_w$(w)

      - length fact in $(W_\lambda, S_\lambda)$

    - tricky bit: induction step is subtle, must take into account both $s < S_\lambda$

      and $S < S_\lambda$, partial order given by $(W_\lambda, S_\lambda)$ may differ

      from partial order given by $(W, S)$.

    Stay tuned for more details...