Our favourite theorem of last semester: Beilinson-Bernstein localisation theorem

\[ G \supset B, \quad g \text{ semisimple Lie algebra} \]

\[
\Gamma : \text{D}(G/B)_\lambda \text{-mod} \xrightarrow{\sim} (g\text{-mod})_{\lambda_\alpha} : \Delta_\lambda
\]

Very general

\[ \Rightarrow \text{can use this to study essentially all } g\text{-modules (e.g. category } \mathcal{O}, \text{ (g,k)-modules, etc.)} \]

Last week Yuping told us about affine Kac-Moody Lie algebras and their rep'n theory.

Q: Do we have a Beilinson-Bernstein-type equivalence in this setting?

A: Not quite. ☹
Why? essential reason:
- $Z(g) \subset U(g)$ is "large" (H.C iso: $Z(g) \cong S(g)^W$) $\Rightarrow$ action tells us a lot
- $Z(\tilde{g}) \subset U(\tilde{g})$ is "small" ($Z(\tilde{g}) = \mathbb{C}C\tilde{e}$ (central extension)) $\Rightarrow$ action only tells us the level

More precisely: Why can we use D-modules on geometric spaces to study Lie algebra reps?

Generally:

- $G \underset{\text{lie group}}{\xrightarrow{\text{differential}}} \mathfrak{g} \xrightarrow{\text{vector fields on } X} U(\mathfrak{g}) \xrightarrow{\text{get alg}} \Gamma(X, \mathcal{D}_X)$

$\Rightarrow$ get a map $\Gamma: \mathcal{D}_X$-modules $\rightarrow \mathfrak{g}$-modules

There's no reason to expect $\Gamma$ to be an equivalence of categories.

The miracle of BB localization: $G \underset{\text{BB}}{\times} X = G / B$

- The map $U(\mathfrak{g}) \rightarrow \Gamma(G / B, \mathcal{D}_{G / B})$ factors over $U(\mathfrak{g}) / (Z(\mathfrak{g}))_0 \rightarrow \Gamma(\mathcal{D}_{G / B})$

- Take associated graded of degree filter:

$$\text{Gr LHS} = \frac{S(\mathfrak{g})}{(S(\mathfrak{g})^W)_{\geq 1}} \quad \text{Gr RHS} = \frac{\Gamma(T^*G / B, \Omega_{T^*G / B})}{\Gamma(W, \Omega_W)}$$

Source: (Kostant)

$\Rightarrow$ $U(\mathfrak{g}) / (Z(\mathfrak{g}))_0 \overset{\text{isomorphism}}{\rightarrow} \Gamma(\mathcal{D}_{G / B})$
This works b/c $\mathfrak{z}(\mathfrak{g})$ is "just the right size" to make $\mathcal{U}(\mathfrak{g})/(\mathfrak{z}(\mathfrak{g})) \simeq \Gamma(G/B, \mathscr{D}_{G/B})$.

So in the special case of the homogeneous space $X = G/B$,

$$\Gamma: \mathcal{D}_{G/B}\text{-mod} \sim\rightarrow \mathcal{U}(\mathfrak{g})/(\mathfrak{z}(\mathfrak{g}))\text{-mod}$$

is an equivalence.

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For a semisimple Lie algebra, we can find a geometric space (the flag variety) s.t. we can study essentially all Lie algebra reps as global sections of D-modules on this space.

Now let $\hat{\mathfrak{g}} = \text{affine Lie algebra (central extension of } \mathfrak{g}(\mathfrak{t})) \text{ by } \mathfrak{c}$

- We have an affine flag variety: $\mathbb{F}_G: = \hat{\mathbb{G}} / \mathcal{I}$

- We have an action $\hat{\mathbb{G}} \curvearrowright \mathbb{F}_G$

  $\Rightarrow$ get map $\mathcal{U}(\hat{\mathfrak{g}}) \rightarrow \Gamma(\mathcal{D}_{\mathbb{F}_G})$

  factors through $\mathcal{U}(\hat{\mathfrak{g}})/(C(C(c))) \uparrow \Gamma(\mathcal{D}_{\mathbb{F}_G})$

  (I think...)

  

  no reason to suspect isomorphism! $\mathfrak{z}(\mathfrak{g})$ "too small" $\Rightarrow \mathcal{U}(\mathfrak{g})/(C(C(c)))$ "too big"

---

For an affine Lie algebra, we don't have a nice geometric space s.t. all $\hat{\mathfrak{g}}$- reps are global sections of D-modules on this space.
No! Still have lots of maps \( \Gamma : \mathcal{D}\text{-modules on some space } X \rightarrow \mathcal{O}\text{-modules} \) which we can use to study \( \mathcal{O}\text{-modules} \), we just don't expect them to be equivalences.

Some useful spaces:

- affine Grassmannian \( \mathcal{G}(\mathfrak{t})//G^\mathfrak{t} \)
- affine flag variety \( \mathcal{G}(\mathfrak{t})//I \)
- enhanced affine flag variety \( \mathcal{G}(\mathfrak{t})//I_u \) (analogue to base affine space)

Goals for these lectures:

1) precisely define these geometric objects: Kac-Moody groups, their flag varieties, simplifications when \( \mathcal{G} \) is affine, ind-scheme structure, \( \mathcal{D}\text{-modules on ind-schemes} \)
   
   resource: Kumar, Kac-Moody groups, their flag varieties, and representation theory

2) In certain settings, if we add enough adjectives to either side, we do get equivalences of categories. I'll clarify what's known and what's conjectured about these equivalences

   resource: Frenkel-Gaitsgory papers (then one paper...), Kashiwara-Tanisaki

3) Sketch the proof of one such equivalence (due to Kashiwara-Tanisaki) which is used in the proof of the negative level KL conjecture Yaping stated last week.

   resource: Frenkel-Gaitsgory, "Localization of \( \mathcal{O}\text{-modules on the affine Grassmannian}"
Kac-Moody groups

- General idea: Given a Kac-Moody Lie algebra \( \hat{\mathfrak{g}} \), build a group \( \hat{G} \) s.t. \( \text{Lie} \hat{G} = \hat{\mathfrak{g}} \)

- Strategy: Use triangular decomposition of \( \hat{\mathfrak{g}} = n^- \oplus \hat{\mathfrak{h}} \oplus n^+ \) to construct a bunch of subgroups of \( \hat{G} \)
  - Build \( \hat{G} \) as their amalgamated product ("built from \( SL_2 \)'s")
  - \( \hat{G} \) + subgroups used to build it form a (topological) Tits system
  \[ \Rightarrow \text{get lots of nice properties of } \hat{G} \text{ for free: Bruhat decomposition, Weyl group, standard parabolics, CW structure on } \hat{G}/B, \text{etc} \]

- Special case: \( \hat{\mathfrak{g}} \) affine

  - \( \hat{G} \) can be realised as a central extension by \( C^\times \) of the loop group \( \hat{G}(C(t)) \)
  - Can use this to realise \( \hat{SL}_n/B \) as a collection of lattices in \( C((t)) \)
    \[ \Rightarrow \text{can use this to explicitly see ind-scheme structure} \]

We'll do it all somewhat carefully.
A pair \((G, \mathcal{F})\), where \(\mathcal{F}\) is a family of normal subgroups s.t. \(\forall N \in \mathcal{F}\), \(G/N\) is an affine algebra group is a pro-group if

(a) If \(N_i, N_j \in \mathcal{F}\), then \(N_i \cap N_j \in \mathcal{F}\).

(b) If \(N \in \mathcal{F}\), then a normal subgroup \(N_2 \supset N_1\) is in \(\mathcal{F}\) if \(N_2/N_1\) is a closed subgroup of \(G/N\) (normal).

(c) If \(N_i, N_j \in \mathcal{F}\) are s.t. \(N_i \subset N_j\), then the quotient map \(\pi_{N_i,N_j}: G/N_i \rightarrow G/N_j\) is a morphism of \(k\)-algebraic groups.

(d) The natural homomorphism \(\gamma: G \rightarrow \varprojlim_{N \in \mathcal{F}} G/N\) is bijective (\(\mathcal{F}\) is a directed set via \(\preceq\).

\(N_1 \preceq N_2\) iff \(N_1 \supset N_2\).

Examples:

- Any algebraic group \(G\) w/ \(\mathcal{F} = \{\text{All normal closed subgroups}\}\) is a pro-group.

- \(\mathcal{F}\) is the set of algebraic groups \(G = \prod_{i \in I} G_i\) w/ \(\mathcal{F} = \{N \subset G\} \text{ s.t. } \prod_{i \in I} \pi_i(N) \subset \prod_{i \in I} G_i\) closed.

  \(\pi_j: \prod_{i \in I} G_i\) factors projection onto 1st \(j\) factors.
Let $S$ be a Lie algebra and $F$ a family of ideals of $S$ of finite codimension. The pair $(S, F)$ is a pro-Lie algebra if the following axioms are satisfied:

(a) if $a, a_1 \in F, a \cap a_2 \in F$

(b) if $a, a_1 \in F$ and $a_1 \supseteq a_2$ is an ideal, then $a_2 \in F$

(b) the canonical Lie algebra homomorphism $\varphi: S \to \varprojlim_{a \in F} \frac{S}{a}$ is an isomorphism. ($F$ has partial order by $\triangleright$)

Consequence: $\bigcap_{a \in F} a = 0$

Examples:

1) A f.d. Lie algebra $S$ is a pro-Lie algebra with $F = \{\text{all ideals of } S\}$

2) $S_i, i \in I$ family of f.d. Lie algebras w/ sub Lie algebra $\varphi_i: S_i \to S_i, \forall i$. Then

$$S = \varprojlim S_i$$

is a pro-Lie algebra w/ $F = \{\text{id} \subset \text{ideal of } S\}, \forall i \in I$: $S \to S$

Pro-groups have pro-Lie algebras:

- $(S, F)$ pro-group, for $N \in F$, build $S_N = \text{Lie } S/N$

- For $N, N_2 \in F$, have $\varphi_{N, N_2}: S/N_1 \to S/N_2$, differentiate $\tilde{\varphi}_{N, N_2}: S_N \to S_{N_2}$, get inverse system

$$S = \varprojlim_{N \in F} S_N$$

is a pro-Lie algebra, $\text{Lie } S$
A pro-group $S$ is pro-unipotent if $S/N$ is unipotent $\forall N \in S$.

A pro-Lie algebra $S$ is pro-nilpotent if $S/\alpha$ is nilpotent $\forall \alpha \in S$.

**Theorem:** \[ \text{category of pro-unipotent pro-groups} \leftrightarrow \text{category of pro-nilpotent pro-Lie algebras} \]

Let us build pro-groups from pro-algebras - it's what we'll use to construct the pieces of Kac-Moody groups.

Example: \( \hat{\mathfrak{g}} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{f} \) Kac-Moody Lie algebra, triangular decomposed given by root spaces.

- \( \mathfrak{n} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \) \( \rightarrow \) \( \hat{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \)

- Lower central series:
  \( \mathfrak{n} \supset [\mathfrak{n}, \mathfrak{n}] \supset [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \supset \cdots \supset \mathfrak{n}(k) = [\mathfrak{n}, \cdots [\mathfrak{n}, \mathfrak{n}] \cdots] \)
  - \( \mathfrak{n}(k) = 0 \) if \( \hat{\mathfrak{g}} \) is infinite, so \( \hat{\mathfrak{n}} = (\mathfrak{n}) \) is not nilpotent.
  - But \( \mathfrak{n}(k) = 0 \), so it is pro-nilpotent.

- Get inverse system:
  \[ \hat{\mathfrak{n}} / [\mathfrak{n}, \mathfrak{n}] \leftarrow \hat{\mathfrak{n}} / [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \leftarrow \cdots \]
  \( \hat{\mathfrak{n}} \rightarrow \lim_{\kappa} \hat{\mathfrak{n}} / \mathfrak{n}(k) = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \) is an iso.

\( \Rightarrow \hat{\mathfrak{n}} \) is a pro-nilpotent pro-Lie algebra.

- Compare to \( \mathfrak{n} \):
  - Lower central series gives same inverse system: \( \hat{\mathfrak{n}} / [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \mathfrak{n} / [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \)
  - But \( \mathfrak{n} \rightarrow \lim_{\kappa} \mathfrak{n} / \mathfrak{n}(k) \) is not an iso.

This is why we need to complete! To give structure of a pro-object.

Theorem \( \Rightarrow \exists \Pi \) pro-group s.t. \( \text{Lie} \Pi = \hat{\mathfrak{n}} \)
The next ingredient: Tits Systems

You've all seen this defn, but I'll remind you for good measure:

\[ \text{defn: A Tits system } \quad \text{or BN pair} \quad \text{is a quadruple } \quad (G, B, N, S) \quad \text{satisfying four axioms:} \]

(\text{BN}_1) \quad B \cap N \leq N \quad \text{and } S \text{ generates } \quad W := N^B N \quad \text{Weyl group of Tits system} \]
(\text{BN}_2) \quad B \text{ and } N \text{ generate } G \text{ as a group} \]
(\text{BN}_3) \quad \text{For any } s \in S, \quad sB^w \subseteq B^w \]
(\text{BN}_4) \quad \text{for } w \in W \text{ and } s \in S, \quad (s)(w)(s) \subseteq (w)(s) \cup (sw), \quad \text{where } \quad (w) := B^w B \text{ for } w \in W \text{ and } \text{rep} \]

Any subgroup \( P \) such that \( B \leq P \leq G \) is a standard parabolic subgroup of \( G \). Any subgroup \( Q \leq G \) conjugate to a standard parabolic subgroup is a parabolic subgroup.

- Of course the classic example is \( (G, B, N, S) \)

- Also have a notion of a refined Tits system \( (G, N, U_+, U_-, H, S) \), where \( G = (N, U_+), \quad H \leq N, \quad S \cap N/H \)

and a bunch of axioms are satisfied. Refined Tits systems give Tits systems \( w B = U \cdot H \).
Properties of a Tits system $(G, B, N, S)$:

- All sets of $S$ are order 2.
- For any subset $Y \subseteq S$, $P_Y := B \cdot W \cdot Y \cdot B$ is a subgroup of $G$.
- (Bruhat decomposition) We have the disjoint unions
  \[ G = \bigsqcup_{w \in W} B \cdot w \cdot B, \quad \text{for any } Y \subseteq S \quad G = \bigsqcup_{w \in W} P_Y \cdot w \cdot P_Y, \]
  \( (s) (sw) = \begin{cases} 
  (sw) & \text{if } l(sw) > l(w) \\
  (w) (sw) & \text{if } l(sw) \leq l(w) 
\end{cases} \)
- For any reduced decomposition $w = s_1 \cdots s_p$ with $s_i \in S$, $(s_i) < (s_i w)$. Moreover, $(B, wBw^{-1}) = (sw)$.

- For $Y, Y' \subseteq S$, $P_Y = P_{Y'}$ iff $Y = Y'$. Furthermore, for any standard parabolic $P$, $\exists Y$ s.t. $P = P_Y$.
- For any parabolic subgroup $P \subseteq G$, $N(P) = P$. Moreover, two standard parabolics $P, Q$ are conjugate implies $P = Q$.
- $G$ is the amalgamated product of its subgroups $\bigsqcup_{s \in S} N, P_s; s \in S$.

$\ast$ Conversely, if we start with $\bigsqcup_{s \in S} L, P_s; s \in S$ satisfying a list of properties similar to the ones above $S$, then $G = \text{amalgamated product of } \bigsqcup_{s \in S} N, P_s; s \in S$ forms a Tits system $(G, B, N, S)$. $\ast$

Can use this to build groups satisfying all the nice properties of Tits systems.
Finally, can use all these ingredients to construct Kac—Moody groups:

- Let $\mathfrak{g} = \mathfrak{g}(A)$ Kac—Moody Lie algebra, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ decomposed given by root spaces

- Complete $\mathfrak{h} \rightarrow \hat{\mathfrak{h}} = \prod_{\alpha \in \Phi} \mathfrak{g}_\alpha$ is a pro-nilpotent pro-Lie algebra (as we showed earlier)

  $\Rightarrow \exists$ corresponding pro-unipotent pro-group $U$ with $\text{Lie } U = \hat{\mathfrak{h}}$

- Let $\hat{U}_i = \prod_{\alpha \in \Phi} \mathfrak{g}_\alpha \subset \hat{\mathfrak{h}}$, and $U_i$ corresponding pro-group with $\text{Lie } U_i = \hat{U}_i$

- Fix an integral form $\mathfrak{h}_\mathbb{Z}$ of $\mathfrak{h}$ (i.e., finitely many $\mathbb{Z}$-submodule satisfying $\mathfrak{h}_\mathbb{Z} = \mathfrak{h}$

- Set $T := \text{Hom}_\mathbb{Z}(\mathfrak{h}_\mathbb{Z}, \mathbb{C}^*)$, then $\mathfrak{h}_\mathbb{Z}$ gives $T \mathfrak{g}$, $T \mathfrak{g} U$, $T \mathfrak{g} U_i$; $b_\mathbb{Z} / Z_{\mathbb{C}^*}Z_{\mathbb{C}^*}$ is torsion-free

- Define $B = U K T$, acquires natural pro-group structure

- Set $N = \langle T, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_r \rangle$ where $\tilde{s}_i$ are formal symbols satisfying $\tilde{s}_i : \tilde{s}_i = \tilde{s}_i (t)$

- Let $G_i = \text{connected reductive group}$ with $\text{Lie } G_i = \mathfrak{g}_i \oplus \mathfrak{h} \oplus \mathfrak{c}_i$

- can show $G_i \cap U_i \sim P_i := U_i K G_i$

Define the Kac—Moody group $G$ of $\mathfrak{g}$ is the amalgamated product of $(N, P_i : i = 1, \ldots, r)$
Nice consequences of this construction:

**Theorem:** \((G, B, N, S)\) is a Tits system (where \(S = \{S_T: T \in N/T\})

\[ G = \bigsqcup B a B \]

- One parameter subgroups: for \(d \in \Delta_{se}, \dim \Delta_{se} = 1\), define \(U_d = \text{Exp}(C x_d)\)
- Define \(U^- = \langle U_d : d \in \Delta_{se} \rangle \subset G\) \(\leftarrow\) Notice lack of symmetry in defn of \(U\) and \(U^-\)!

**Theorem:** \((G, N, U, U^-, T, S)\) is a refined Tits system

\[ G = \bigsqcup U^- \bigsqcup U \]

- Can define pro-representations of both \(G\) and \(\hat{G} = \pi \otimes \mathbb{R}\) (completion of \(G\))

**Theorem:** There is an equivalence of categories

\[
\begin{align*}
\text{category of} & \quad \text{category of pro-reps of } \hat{G} \\
\text{pro-representations} & \quad \text{s.t. } b\text{-action integrates to a locally finite algebraic } \\
of G & \quad T\text{-module structure}
\end{align*}
\]
Flag varieties (I'll be brief/vague here so I can be more explicit in the affine case, which is what we really care about.)

- The homogeneous space $G/B$ can be given the structure of a projective ind-variety $\text{such that}$ $U \cdot X = X$ 

- The Schubert varieties $X_w = \bigsqcup_{u \leq w} Bv_B/B$ are closed projection subvarieties $\Rightarrow$ all $X_w$ are projective and $X_w \cap X_{w'}$ closed.

- A sketch of this construction:
  - $x \in B^* \text{ dominant integral, } V(x) \text{ integrable highest wt $g$-module, } G \cdot V(x) \text{ by previous theorem}$
  - orbit through highest weight vector in $\text{IP}(V(x))$ gives map $i_v(x) : G/B \rightarrow \text{IP}(V(x))$: lines in $V(x)$

- $G/B$ has filtration by $X_n = \bigcup_{v \in B} Bv_B/B$

- Show $i_v(x)(X_n)$ and $i_v(x)(X_w)$ are closed in $\text{IP}(V(x))$ for any $n \geq 0$

$\Rightarrow$ gives $G/B$ a (unique) projective ind-variety structure w/ filtration $\bigcup_{n \geq 0} X_n$ such that $i_v(x)$ is closed embedding.

- A priori depends on $x$, but can show that for $g$ symmetric, $G$ birational onto $G/B(x)$ for any dominant integral $x \in g^*$

- The $T$-fixed points in $G/B$ are $\bigcup_{\omega \in W} B\overline{B} \cdot \omega \in G/B$.

- $U \cdot G/B$ and $U \cdot \text{id} \subset G/B$ is open, inherits from $G/B$ ind-variety structure making it into an affine ind-group.
For $\mathfrak{g}$ affine, we can realise all of this more explicitly.

- Let $\mathfrak{g} = \text{p.d. simple Lie algebra}$, $\mathfrak{g} = \text{corresponding affine Lie algebra}$ (central extension of loop)
  \[ \mathfrak{g} : \text{connected, simply connected algebra} \quad G = \text{Kac–Moody group w/ Lie } \mathfrak{g} = \mathfrak{g} \]

**Definition** For $R = \text{associative } \mathbb{C}\text{-algebra}$, $\hat{G}(R) = \text{set of } R\text{-rational points of } \hat{G}$; i.e. set of $\mathbb{C}\text{-algebra homs}$ $C[G] \to R$. $\hat{G}(R)$ has canonical group structure.

- E.g. if $\hat{G} = SL_n(\mathbb{C})$, $\hat{G}(R) = SL_n(R)$
- For $R = \mathbb{C}(\!(t)\!) \text{ Laurent series (i.e. infinitely many powers of } t)$, this yields the loop group of $\hat{G}$:
  \[ L(\hat{G}) = \hat{G}(\mathbb{C}(\!(t)\!)) \]

Aside:

Why "loop group"? Can be realised as group of smooth maps $S^1 \to \hat{G}$ "loops in $\hat{G}$" history: arise in 2D quantum field theory

"In fact, it is not much of an exaggeration to say that the mathematics of two-dimensional quantum field theory is almost the same thing as the representation theory of loop groups." Segal, Loop groups
Extend $\mathcal{L}(\hat{G})$ by "exp α":
- consider group hom $\gamma: C^* \rightarrow \text{Aut}(\mathcal{L}(\hat{G}))$, $\gamma(z)(p\lambda) = p(z\lambda)$ "loop rotation"
- get $\gamma_\alpha: C^* \rightarrow \text{Aut}(\mathcal{L}(\hat{G}))$
- Define $\mathcal{L}(\hat{G}) := C^* \times \mathcal{L}(\hat{G})$

Relationship between $\mathcal{L}(\hat{G})$ and $\hat{G}$:
- Theorem: ∃ group hom $\psi: G \rightarrow \mathcal{L}(\hat{G})/\mathcal{L}$ s.t. $\psi$ is surjective and $\ker\psi = \text{center of } G$
  ⇒ We can study $\mathcal{L}(\hat{G})$ instead of $G$ and we basically lose no information

Subgroups of $\mathcal{L}(\hat{G})$: $\mathcal{L}(\hat{G})$

$C^* \times \hat{G}(\mathcal{L}(\hat{G})) \xrightarrow{\text{ev}_0} C^* \times \hat{G}$

$\hat{T} := \text{Hom}_\mathbb{Z}(\hat{G}^*, C^*) \subset \hat{G}$ maximal torus

$\hat{T} := C^* \times \hat{O} \subset \mathcal{L}(\hat{G})$

Then: $\psi$ induces bijection

$G/\mathcal{L} \cong \mathcal{L}(\hat{G})/\mathcal{L}$
A quick remark on central extensions:

\[ \tilde{I}(\tilde{G}) \]

\[ \mathbb{C}^* \times \tilde{G} \xrightarrow{ev_0} \mathbb{C}^* \times \tilde{G} \]

Flag varieties don't see the difference:

"the affine flag variety"

\[ \tilde{I}(\tilde{G}) / \tilde{I} = \tilde{G}(t) / \tilde{I} \]

What's the point of introducing the central extension?

- Both \( \tilde{I}(\tilde{G}) \) and \( \tilde{G}(t) \) act on these homogeneous spaces. There are certain line bundles which are not \( \tilde{G}(t) \)-equivariant, but are \( \tilde{I}(\tilde{G}) \)-equivariant. So it's good to remember the \( \tilde{I}(\tilde{G}) \)-action.
Lattice model for SLn:

**Definition:** A lattice in $\mathbb{C}(t)^n$ is a $\mathbb{C}[t]$-submodule $\mathcal{L} \subset \mathbb{C}(t)^n$ s.t. $\mathcal{L} \otimes \mathbb{C}[t] / \mathcal{L} \cong \mathbb{C}(t)^n$.

- the standard lattice is $\mathcal{L}_0 = \mathbb{C}[t]^n$

- Have natural action $GL_n(\mathbb{C}(t)) \rtimes \{ \text{lattices in } \mathbb{C}(t)^n \}$
  - transitively
  - $\text{Stab}_{GL_n(\mathbb{C}(t))} \mathcal{L}_0 = GL_n(\mathbb{C}[t])$

We can put an ind-scheme structure on $\{ \text{lattices} \}$ as follows:

- Set $Gr^N = \{ \mathcal{L} \mid t^n \mathcal{L}_0 \supset \mathcal{L} \supset t^n \mathcal{L}_0 \}$ lattices whose poles and zeroes have order $\leq N$

  - Note that:
    
    $Gr^N \leftrightarrow \{ \text{t-stable subspaces of } t^{-N} \mathcal{L}_0 / t^{-N} \mathcal{L}_0 \}$
    
    $\mathcal{L} \rightarrow \text{image of } \mathcal{L} \text{ in } t^{-N} \mathcal{L}_0 / t^{-N} \mathcal{L}_0$
    
    $\psi^{-1}(M) \leftarrow M$

    here

    $\psi : t^{-N} \mathcal{L}_0 \rightarrow t^{-N} \mathcal{L}_0$
What does $t^n L_0 / t^n L_0$ look like?

- Well, as a vector space it is $2N$ copies of $C^n$, with a $t$-action given by:

$$
\begin{array}{c}
\vdash C^n \oplus C^n \oplus \ldots \oplus C^n
\end{array}
\cong C^{2nN}
$$

- The subspace corresponding to $L_0$ is $0 \oplus 0 \oplus \ldots \oplus 0 \oplus C^n \oplus \ldots \oplus C^n$ (a $nN$-dim subspace)

$$
\text{Gr}^N \xrightarrow{\text{Gr}} \text{Grass} (C^{2nN}) = \bigcup_{d \geq 0} \text{Grass} (C^{2nN}, d)
$$

- Give $\text{Gr}^N$ the necessary topology to make this embedding closed.

$$
\Rightarrow \text{Gr}^N \text{ gains the structure of a finite-dim projective variety}
$$

So $\text{Gr} = \bigcup_{N \geq 0} \text{Gr}^N$ gains the structure of a projective ind-variety.

**Theorem:** This agrees w/ the ind-variety structure I sketched earlier.
For $\text{Sl}_2$ we can be even more explicit:

$$\text{Gr}_{\text{Sl}_2} = \mathcal{E}\text{lattices in } \mathfrak{sl}(2)^2 \text{ of det } \pm 1$$

represent a lattice $\mathcal{L}$ by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

minimal occurs in 2nd row or 5th row

- let $-m = \text{minimal valuation (}m\geq 0\text{)}$

(i.e. $m = \text{max order of pole in } a,b,c,d$)

$$\begin{cases} \begin{pmatrix} t^{-m} & f \\ 0 & t^m \end{pmatrix} : f = f, t^{-m} + \cdots + t^{-m} \\ \mathcal{C}^{2m} \quad \text{cell} \quad \mathcal{C}^{2m} \end{cases}$$

$$\begin{cases} \begin{pmatrix} t^{-m} & 0 \\ g & t^m \end{pmatrix} : g = g, t^{-m} + \cdots + t^{-m} \\ \mathcal{C}^{2m-1} \quad \text{cell} \quad \mathcal{C}^{2m-1} \end{cases}$$

Get cell decomposition

$$\text{Gr}_{\text{Sl}_2} = \mathcal{C}^0 \sqcup \mathcal{C}^1 \sqcup \mathcal{C}^2 \sqcup \mathcal{C}^3 \sqcup \cdots$$

and

$$\overline{\mathcal{C}} = \coprod_{i=1}^n \mathcal{C}^i$$

Schubert cells of affine Grassmannian
(parametrized by cocharacter lattice)

Surely I’ve ran out of time, but can also do this for any group $G \to \text{SL}_n$, and can consider

$$\mathcal{E}(\text{lattice, } \mathfrak{l} \subset \mathfrak{g})^2$$

as a model for $\text{Gr}_{\text{Sl}_2} = \text{Sl}_2(\mathcal{E})/I$. More next week!