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EQUIVARIANT VECTOR BUNDLES ON QUANTUM HOMOGENEOUS SPACES

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Abstract. The notion of quantum group equivariant homogeneous vector bundles on quantum homogeneous spaces is introduced. The category of such quantum vector bundles is shown to be exact, and its Grothendieck group is determined. It is also shown that the algebras of functions on quantum homogeneous spaces are noetherian.

1. Introduction

Quantum homogeneous spaces are concrete examples of noncommutative geometries which are interesting to several areas in mathematics and physics. The simplest quantum homogeneous spaces such as the quantum spheres have been particularly well investigated, see, e.g., [7, 4] and references therein. In the joint publication [5] by one of us with Gover and also [12], quantum homogeneous spaces were applied to study a geometric representation theory for quantum groups and quantum supergroups. A quantum analogue of the Bott-Borel-Weil theory [2] as envisaged in [8] emerged [5, 12] from the study, which in particular placed the work of Andersen, Polo and Wen [1] in the setting of noncommutative geometry.

If $\mathrm{U}_q(\mathfrak{l})$ is a reductive subalgebra of the quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{g})$, the algebra $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ of $\mathrm{U}_q(\mathfrak{l})$-invariant functions on the quantum group associated with $\mathrm{U}_q(\mathfrak{g})$ defines a quantum homogeneous space (see Remark 2.2 for further discussion) in the general spirit of noncommutative geometry [3]. A quantum vector bundle on the quantum homogeneous space is defined by specifying the space of sections M. Here it is important that M must be a finitely generated projective module over $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$. From the viewpoint of representation theory, the interesting quantum homogeneous vector bundles are those admitting actions of a quantum group, which are analogues of equivariant vector bundles (see, e.g., [10]) in classical differential geometry. The purpose here is to classify such noncommutative vector bundles.

The algebra $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ will be shown to be noetherian. We introduce a notion of $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-modules, and show that the category $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$ of finitely generated $\mathrm{U}_q(\mathfrak{g})$-equivariant projective $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-modules is an exact category. Objects of $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$ are regarded as $\mathrm{U}_q(\mathfrak{g})$-equivariant quantum homogeneous vector bundles, which are shown to correspond to isomorphism classes of finite dimensional representations of $\mathrm{U}_q(\mathfrak{l})$. The precise statement of this result is given in Theorem 3.9.

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The natural framework for understanding results in this letter is a possible quantum group equivariant algebraic K-theory for the algebra $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ of functions on the quantum homogeneous space. We allude to this very briefly at the end.

2. Quantum homogenous spaces

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra of rank r with the set of simple roots $\Pi = \{\alpha_i | i = 1, 2, \ldots, r\}$. We denote by $\mathcal{P}$ the set of the integral weights, and by $\mathcal{P}_+$ the set of the integral dominant weights of $\mathfrak{g}$. Fix $q \in \mathbb{C}^*$, which is non-zero and is not a root of 1. The quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ will be presented with the standard generators $\{e_i, f_i, k_i^{\pm 1} \mid i = 1, \ldots, r\}$ and relations (see, e.g., [1, 6]). As is well known, $\mathrm{U}_q(\mathfrak{g})$ has the structure of a Hopf algebra. We denote by Δ, ϵ and S the co-multiplication, co-unit and antipode respectively.

We shall only consider the finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-modules of type-$(1, \ldots, 1)$. It follows from standard facts in Hopf algebra theory [9] that the matrix elements [5] of the $\mathrm{U}_q(\mathfrak{g})$-representations on such modules span a Hopf subalgebra $\mathcal{A}(\mathfrak{g})$ of the finite dual $\mathrm{U}_q(\mathfrak{g})^\circ$ of $\mathrm{U}_q(\mathfrak{g})$. For convenience, we shall also denote the co-multiplication and the antipode of $\mathcal{A}(\mathfrak{g})$ by Δ and S respectively.

There exist two natural actions R and L of $\mathrm{U}_q(\mathfrak{g})$ on $\mathcal{A}(\mathfrak{g})$ [5], which correspond to the left and right translations in the context of Lie groups. The actions are respectively defined by

$$R_x f = \sum_{(f)} f_{(1)} < f_{(2)}, x >, \qquad L_x f = \sum_{(f)} < f_{(1)}, S(x) > f_{(2)}$$

for all $x \in \mathrm{U}_q(\mathfrak{g})$ and $f \in \mathcal{A}(\mathfrak{g})$. Here we have used Sweedler's notation to write the co-multiplication of f as $\Delta(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}$, and also have written $f(x)$ as $\langle f, x \rangle$ for any $f \in \mathrm{U}_q(\mathfrak{g})^\circ$ and $x \in \mathrm{U}_q(\mathfrak{g})$. The two actions commute, and $\mathcal{A}(\mathfrak{g})$ forms a $\mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})$ module under the action $L \otimes R$.

Let Θ be a subset of $\{1, 2, \ldots, r\}$. We denote by $\mathrm{U}_q(\mathfrak{l})$ the reductive Hopf subalgebra of $\mathrm{U}_q(\mathfrak{g})$ generated by the elements of $\{k_i^{\pm 1} \mid 1 \leq i \leq r\} \cup \{e_j, f_j \mid j \in \Theta\}$. The Hopf algebra embedding $\iota : \mathrm{U}_q(\mathfrak{l}) \longrightarrow \mathrm{U}_q(\mathfrak{g})$ induces a Hopf algebra map $p : \mathrm{U}_q(\mathfrak{g})^0 \longrightarrow \mathrm{U}_q(\mathfrak{l})^0$, which is defined for any $f \in \mathrm{U}_q(\mathfrak{g})^0$ by $\langle p(f), u \rangle = \langle f, \iota(u) \rangle$, $\forall u \in \mathrm{U}_q(\mathfrak{l})$. Denote by $\mathcal{A}(\mathfrak{l})$ the Hopf subalgebra $p(\mathcal{A}(\mathfrak{g}))$ of $\mathrm{U}_q(\mathfrak{l})^0$.

Recall that any right $\mathcal{A}(\mathfrak{l})$-comodule V has a natural left $\mathrm{U}_q(\mathfrak{l})$-module structure. Since all finite dimensional $\mathrm{U}_q(\mathfrak{g})$-modules are completely reducible with respect to $\mathrm{U}_q(\mathfrak{l})$, all $\mathcal{A}(\mathfrak{l})$-comodules are semi-simple. There exists a unique left and right invariant integral $\int : \mathcal{A}(\mathfrak{l}) \longrightarrow \mathbb{C}$ such that $\int \mathbf{1} = 1$, where $\mathbf{1}$ is the identity element of $\mathcal{A}(\mathfrak{l})$.

We denote by $\mathrm{U}_q(\mathfrak{l})$-**mod** the category of finite dimensional left $\mathrm{U}_q(\mathfrak{l})$-modules, which arise from right $\mathcal{A}(\mathfrak{l})$-comodules.

Definition 2.1. *Define* $\mathcal{A}(\mathfrak{g}, \mathfrak{l}) := \{f \in \mathcal{A}(\mathfrak{g}) \mid L_x(f) = \epsilon(x) f, \ \forall x \in \mathrm{U}_q(\mathfrak{l})\}$.

Since $\mathrm{U}_q(\mathfrak{l})$ is a Hopf subalgebra of $\mathrm{U}_q(\mathfrak{g})$, it follows from the definition that $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ is a subalgebra of $\mathcal{A}(\mathfrak{g})$. It is important to point out that $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ is a right co-ideal of $\mathcal{A}(\mathfrak{g})$, that is, $\Delta(\mathcal{A}(\mathfrak{g}, \mathfrak{l})) \subset \mathcal{A}(\mathfrak{g}, \mathfrak{l}) \otimes \mathcal{A}(\mathfrak{g})$. This in particular allows us to define a right $\mathcal{A}(\mathfrak{g})$-comodule structure on $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ by $a \mapsto \Delta(a)$, $a \in \mathcal{A}(\mathfrak{g}, \mathfrak{l})$. Equivalently

$\mathcal{A}(\mathfrak{g},\mathfrak{l})$ has a left $\mathrm{U}_q(\mathfrak{g})$-module structure with the action being the restriction of the right translation R. Clearly for any $a, b \in \mathcal{A}(\mathfrak{g},\mathfrak{l})$,

$$x \cdot (ab) = \sum_{(x)} R_{x_{(1)}}(a) \cdot R_{x_{(2)}}(b), \quad \forall x \in \mathrm{U}_q(\mathfrak{g}),$$

and the identity element ϵ of $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ is $\mathrm{U}_q(\mathfrak{g})$-invariant. Thus $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ is a $\mathrm{U}_q(\mathfrak{g})$-algebra.

Remark 2.2. The algebra $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ was constructed in [5]. It was shown there that $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ was the natural quantum analogue of the algebra of functions on G/K for a compact connected Lie group G and a closed subgroup K, where G and K have the Lie algebras $\mathcal{L}ie(G)$ and $\mathcal{L}ie(K)$ with complexifications $\mathfrak{g}$ and $\mathfrak{l}$ respectively.

It was shown in [6, Proposition 9.2.2] that $\mathcal{A}(\mathfrak{g})$ is noetherian (where $\mathcal{A}(\mathfrak{g})$ is denoted by $R = R_q[G]$, see also [6, §9.1.1]). We have the following result.

Theorem 2.3. *$\mathcal{A}(\mathfrak{g},\mathfrak{l})$ is noetherian.*

Proof. Assume the existence of an infinite ascending chain $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ of left ideals in $\mathcal{A}(\mathfrak{g},\mathfrak{l})$, then there is the infinite sequence of left ideals $J_i := \mathcal{A}(\mathfrak{g})I_i$, $i \geq 0$, of $\mathcal{A}(\mathfrak{g})$ such that

$$J_0 \subset J_1 \subset J_2 \subset \cdots .$$

Under the left action $L_{\mathrm{U}_q(\mathfrak{l})}$, $\mathcal{A}(\mathfrak{g})$ decomposes into the direct sum of two submodules $\mathcal{A}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g},\mathfrak{l}) \oplus \mathcal{A}(\mathfrak{g},\mathfrak{l})^\perp$, where $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ comprises trivial 1-dimensional irreducible $\mathrm{U}_q(\mathfrak{l})$-modules only, while $\mathcal{A}(\mathfrak{g},\mathfrak{l})^\perp$ is a direct sum of finite dimensional irreducible $\mathrm{U}_q(\mathfrak{l})$-modules of dimensions greater than 1. If $f \in \mathcal{A}(\mathfrak{g},\mathfrak{l})^\perp$ and $a \in \mathcal{A}(\mathfrak{g},\mathfrak{l})$, then $L_x(af) = aL_x(f)$ and $L_x(fa) = L_x(f)a$ for all $x \in \mathrm{U}_q(\mathfrak{l})$. Hence af and fa belong to $\mathcal{A}(\mathfrak{g},\mathfrak{l})^\perp$, showing that $\mathcal{A}(\mathfrak{g},\mathfrak{l})^\perp$ forms a two-sided $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module under the multiplication in $\mathcal{A}(\mathfrak{g})$.

This in particular implies that

$$J_i \quad = \mathcal{A}(\mathfrak{g},\mathfrak{l})I_i \oplus \mathcal{A}(\mathfrak{g},\mathfrak{l})^\perp I_i \quad = I_i \oplus \mathcal{A}(\mathfrak{g},\mathfrak{l})^\perp I_i.$$

Now $I_i \subsetneq I_{i+1}$, thus $J_i \subsetneq J_{i+1}$, for all i. This way we obtain an infinite ascending chain J_i, $i \geq 0$, of left ideals in $\mathcal{A}(\mathfrak{g})$, contradicting the noetherianity of $\mathcal{A}(\mathfrak{g})$ established in [6, Proposition 9.2.2]. Hence $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ must be left noetherian. In the same way, we can show that $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ is right noetherian, thus is noetherian. □

In view of Remark 2.2, the subalgebra $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ will be refer to as the algebra of functions on a quantum homogeneous space following the terminology of [5]. Adopting the general philosophy of noncommutative geometry [3], we regard the quantum homogeneous space as defined by $\mathcal{A}(\mathfrak{g},\mathfrak{l})$. Also, finitely generated projective modules over $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ will be called quantum homogeneous vector bundles over the quantum homogeneous space.

3. Equivariant quantum homogeneous vector bundles

3.1. Equivariant quantum homogeneous vector bundles. Let M be a left $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module with a structure map $\phi : \mathcal{A}(\mathfrak{g},\mathfrak{l}) \otimes M \longrightarrow M$. We also assume that M is a locally finite left $\mathrm{U}_q(\mathfrak{g})$-module with a structure map $\mu : \mathrm{U}_q(\mathfrak{g}) \otimes M \longrightarrow M$, and denote the corresponding right $\mathcal{A}(\mathfrak{g})$-comodule structure by $\delta : M \longrightarrow M \otimes \mathcal{A}(\mathfrak{g})$. Now $\mathcal{A}(\mathfrak{g}, \mathfrak{l}) \otimes M$ has a natural $\mathrm{U}_q(\mathfrak{g})$-module structure

$$\begin{aligned} \mu' : \mathrm{U}_q(\mathfrak{g}) \otimes (\mathcal{A}(\mathfrak{g}, \mathfrak{l}) \otimes M) &\longrightarrow \mathcal{A}(\mathfrak{g}, \mathfrak{l}) \otimes M, \\ x \otimes (a \otimes m) &\mapsto \sum_{(x)} R_{x_{(1)}}(a) \otimes x_{(2)} \cdot m. \end{aligned}$$

We say that the $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$- and $\mathrm{U}_q(\mathfrak{g})$-module structures of M are compatible if the following diagram commutes

$$\begin{array}{ccc} \mathrm{U}_q(\mathfrak{g}) \otimes \mathcal{A}(\mathfrak{g}, \mathfrak{l}) \otimes M & \stackrel{\mathrm{id}\otimes\phi}{\longrightarrow} & \mathrm{U}_q(\mathfrak{g}) \otimes M \\ \mu' \downarrow & & \mu \downarrow \\ \mathcal{A}(\mathfrak{g}, \mathfrak{l}) \otimes M & \stackrel{\phi}{\longrightarrow} & M. \end{array} \tag{3.1}$$

In this case, M is called a $\mathrm{U}_q(\mathfrak{g})$*-equivariant* $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$*-module*.

A morphism between two $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-modules is an $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module map which is at the same time also a $\mathrm{U}_q(\mathfrak{g})$-module map. We call a $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module M *finitely generated* if there exists an epimorphism $W \longrightarrow M$ where W is free and of finite rank over $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$. We denote by $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$ the category of finitely generated $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-modules, which is an abelian category.

Let $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$ denote the full subcategory of $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$ with objects having the following property. If P is in $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$, there exist objects Q and W in $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$ with W being free over $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ such that

$$P \oplus Q \cong W.$$

We shall refer to any object in $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$ as a *finitely generated* $\mathrm{U}_q(\mathfrak{g})$*-equivariant projective* $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$*-module*. We shall show in Section 4 that a $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module is projective in $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$ if and only if it belongs to $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$.

Definition 3.1. *Call* $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$ *the category of* $\mathrm{U}_q(\mathfrak{g})$*-equivariant quantum homogeneous vector bundles on the quantum homogeneous space defined by* $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$.

Remark 3.2. The cross product $\mathcal{R}(\mathfrak{g}, \mathfrak{l}) = \mathcal{A}(\mathfrak{g}, \mathfrak{l}) \rtimes \mathrm{U}_q(\mathfrak{g})$ is an associative algebra, and a $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module is nothing else but a locally $1 \otimes \mathrm{U}_q(\mathfrak{g})$-finite left $\mathcal{R}(\mathfrak{g}, \mathfrak{l})$-module.

For any object Ξ of $\mathrm{U}_q(\mathfrak{l})$-**mod**, we define

$$\mathcal{S}(\Xi) := \left\{ \zeta \in \Xi \otimes \mathcal{A}(\mathfrak{g}) \,\middle|\, \sum_{(x)} (x_{(1)} \otimes L_{x_{(2)}})\zeta = \epsilon(x)\zeta, \ \forall x \in \mathrm{U}_q(\mathfrak{l}) \right\}. \tag{3.2}$$

Then $\mathcal{S}(\Xi)$ is a left $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module with the action defined by $b\zeta = \sum v_i \otimes b a_i$ for any $\zeta = \sum v_i \otimes a_i \in \mathcal{S}(\Xi)$ and $b \in \mathcal{A}(\mathfrak{g}, \mathfrak{l})$. We have the following result.

Proposition 3.3. *Let* Ξ *be any object in* $\mathrm{U}_q(\mathfrak{l})$-**mod**. *Then* $\mathcal{S}(\Xi)$ *gives rise to a* $\mathrm{U}_q(\mathfrak{g})$*-equivariant quantum homogenous vector bundle.*

Proof. This can be deduced from [5]. Note that $\mathcal{S}(\Xi)$ forms a left $\mathrm{U}_q(\mathfrak{g})$-module with the action defined by

$$x\zeta = (\mathrm{id}_\Xi \otimes R_x)\zeta = \sum v_i \otimes R_x(a_i),$$

for any $x \in \mathrm{U}_q(\mathfrak{g})$ and $\zeta = \sum v_i \otimes a_i \in \mathcal{S}(\Xi)$. We have

$$x(b\zeta) \quad = \sum v_i \otimes R_x(ba_i) \quad = \sum_{(x)} R_{x_{(1)}}(b)(x_{(2)}\zeta), \quad \text{for } b \in \mathcal{A}(\mathfrak{g}, \mathfrak{l}),$$

that is, the $\mathrm{U}_q(\mathfrak{g})$- and $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-actions on $\mathcal{S}(\Xi)$ render the diagram (3.1) commutative. Hence $\mathcal{S}(\Xi)$ indeed forms a $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module.

Consider the bijection [5]

$$\kappa : M \otimes \mathcal{A}(\mathfrak{g}) \longrightarrow M \otimes \mathcal{A}(\mathfrak{g}), \qquad m \otimes f \mapsto \sum_{(m)} m_{(1)} \otimes f S^{-1}(m_{(2)}), \tag{3.3}$$

with the inverse map given by $\kappa^{-1}(m \otimes f) = \sum_{(m)} m_{(1)} \otimes f m_{(2)}$. We give the following $\mathrm{U}_q(\mathfrak{g})$-module structures to the domain and range of κ respectively: any element $x \in \mathrm{U}_q(\mathfrak{g})$ acts on the domain by $x \cdot (m \otimes f) = m \otimes R_x(f)$, and on the range by $x \cdot (m \otimes f) = \sum x_{(2)} m \otimes R_{x_{(1)}}(f)$ via the opposite co-multiplication $\Delta'(x) = \sum x_{(2)} \otimes x_{(1)}$. The $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-actions on both the domain and the range of κ are given by $a \cdot (m \otimes f) = m \otimes af$. Then direct calculations can show that both the domain and range of κ are $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-modules, and κ is a $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module homomorphism. Now we have [5]

$$\mathcal{S}(M) = \kappa^{-1}(M \otimes \mathcal{A}(\mathfrak{g}, \mathfrak{l})). \tag{3.4}$$

Recall that Ξ can always be embedded in the restriction of some finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module as a direct summand. That is, there exists a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module M and another $\mathrm{U}_q(\mathfrak{l})$-module $\Xi^{\perp}$ such that $M \cong \Xi \oplus \Xi^{\perp}$ as $\mathrm{U}_q(\mathfrak{l})$-module. It immediately follows from (3.4) that $\mathcal{S}(\Xi) \oplus \mathcal{S}(\Xi^{\perp}) \cong M \otimes \mathcal{A}(\mathfrak{g}, \mathfrak{l})$. This completes the proof. □

In view of the proposition, we can extend (3.2) to a functor

$$\mathcal{S} : \mathrm{U}_q(\mathfrak{l})\textbf{-mod} \longrightarrow \mathcal{P}(\mathfrak{g}, \mathfrak{l}), \tag{3.5}$$

which applies to objects of $\mathrm{U}_q(\mathfrak{l})$**-mod** according to (3.2) and sends a morphism f to $f \otimes \mathrm{id}_{\mathcal{A}(\mathfrak{g})}$. Since $\mathrm{U}_q(\mathfrak{l})$**-mod** is semi-simple and $\mathcal{S}(V \oplus W) = \mathcal{S}(V) \oplus \mathcal{S}(W)$ for any direct sum $V \oplus W$ of objects in $\mathrm{U}_q(\mathfrak{l})$**-mod**, the functor $\mathcal{S}$ is exact.

3.2. Classification. Recall [10] that for a compact semi-simple Lie group G, and a closed reductive subgroup K, the G-equivariant vector bundles on the homogeneous space G/K correspond bijectively to the isomorphism classes of finite dimensional K-modules. We shall establish a similar result in the quantum case.

Let $I = \{f \in \mathcal{A}(\mathfrak{g}, \mathfrak{l}) | f(1) = 0\}$, which is a maximal ideal of $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$. For any $x \in \mathrm{U}_q(\mathfrak{l})$ and $a \in I$, $\langle R_x(a), 1\rangle = \epsilon(x)a(1) = 0$, thus I forms a $\mathrm{U}_q(\mathfrak{l})$-algebra under the restriction of R. This in particular implies that for any $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module M, IM is a $\mathrm{U}_q(\mathfrak{l})$-equviaraint $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-submodule of M, since for any $a \in I$ and $m \in M$, we have $x(am) = \sum_{(x)} R_{x_{(1)}}(a) x_{(2)} m \in IM$ for all $x \in \mathrm{U}_q(\mathfrak{l})$.

Given a $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module M, we define

$$M_0 = (\mathcal{A}(\mathfrak{g}, \mathfrak{l})/I) \otimes_{\mathcal{A}(\mathfrak{g}, \mathfrak{l})} M.$$

Note the algebra isomorphism $\mathcal{A}(\mathfrak{g}, \mathfrak{l})/I \cong \mathbb{C}$ given by $a + I \mapsto a(1)$. Thus we shall simply write the image in M_0 of an element $m \in M$ as $1 \otimes m$. Define a $\mathrm{U}_q(\mathfrak{l})$-action on M_0 by

$$\mathrm{U}_q(\mathfrak{l}) \otimes M_0 \longrightarrow M_0, \qquad x \otimes (1 \otimes m) \mapsto 1 \otimes x \cdot m, \tag{3.6}$$

which we now show to be well-defined. For all $x \in \mathrm{U}_q(\mathfrak{l})$, $a \in \mathcal{A}(\mathfrak{g}, \mathfrak{l})$ and $m \in M$, we have (in Sweedler's notation)

$$\begin{aligned} x \cdot (am) &= \sum_{(a),(m)} a_{(1)} m_{(1)} \langle a_{(2)} m_{(2)}, x \rangle \\ &= \sum_{(a),(m),(x)} a_{(1)} m_{(1)} \langle a_{(2)}, x_{(1)} \rangle \langle m_{(2)}, x_{(2)} \rangle \\ &= \sum_{(m)} a m_{(1)} \langle m_{(2)}, x \rangle. \end{aligned}$$

Thus $1 \otimes x \cdot (am) = a(1) \otimes x \cdot m = x \cdot (1 \otimes am)$, proving that the $\mathrm{U}_q(\mathfrak{l})$-action (3.6) is indeed well defined.

To summarise,

Proposition 3.4. *The natural* $\mathrm{U}_q(\mathfrak{l})$*-action on* M *(i.e., the restriction of the* $\mathrm{U}_q(\mathfrak{g})$*-action) descends to the* $\mathrm{U}_q(\mathfrak{l})$*-action* (3.6) *on* M_0.

Note that for any finitely generated $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module M, the $\mathrm{U}_q(\mathfrak{l})$-module M_0 is finite dimensional and in fact is a right $\mathcal{A}(\mathfrak{l})$-comodule (note that $\mathcal{A}(\mathfrak{l}) = p(\mathrm{U}_q(\mathfrak{g})^0) \subsetneqq \mathrm{U}_q(\mathfrak{l})^0$), thus belongs to $\mathrm{U}_q(\mathfrak{l})$-**mod**. Hence we can construct the functor

$$\mathcal{E} : \mathcal{M}(\mathfrak{g}, \mathfrak{l}) \longrightarrow \mathrm{U}_q(\mathfrak{l})\text{-}\mathbf{mod}, \tag{3.7}$$

which sends an object M to M_0, and a morphism f to $\mathrm{id}_{(\mathcal{A}(\mathfrak{g},\mathfrak{l})/I)} \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} f$.

We shall show in Proposition 3.8 that if M is an object of $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$, then $\mathcal{S} \circ \mathcal{E}(M)$ is isomorphic to M. To prove this result, we need some preparations. Now for any object M of $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$, we define the map

$$\psi : M \xrightarrow{\delta} M \otimes_{\mathbb{C}} \mathcal{A}(\mathfrak{g}) \longrightarrow (\mathcal{A}(\mathfrak{g}, \mathfrak{l})/I) \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} M \otimes_{\mathbb{C}} \mathcal{A}(\mathfrak{g}), \tag{3.8}$$

where δ is the $\mathcal{A}(\mathfrak{g})$-comodule map of M, and the second map is the obvious one. Introduce a $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module structure on the range of ψ defined for any element $\zeta = 1 \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} m \otimes_{\mathbb{C}} f$ by

$$\begin{aligned} x\zeta &= 1 \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} m \otimes_{\mathbb{C}} R_x(f), \quad x \in \mathrm{U}_q(\mathfrak{g}), \\ a\zeta &= 1 \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} m \otimes_{\mathbb{C}} af, \quad a \in \mathcal{A}(\mathfrak{g}, \mathfrak{l}). \end{aligned}$$

Lemma 3.5. *The map* ψ *is a* $\mathrm{U}_q(\mathfrak{g})$*-equivariant* $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$*-module homomorphism. Furthermore,* $\psi(M) \cong \mathcal{S}(M_0)$ *for any object* M *in* $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$.

Proof. For all $a \in \mathcal{A}(\mathfrak{g}, \mathfrak{l})$, $x \in \mathrm{U}_q(\mathfrak{g})$ and $m \in M$, we have

$$\begin{aligned}
\psi(am) &= \sum_{(a)(m)} 1 \otimes a_{(1)}m_{(1)} \otimes a_{(2)}m_{(2)} \\
&= \sum_{(a)(m)} 1 \otimes m_{(1)} \otimes \epsilon(a_{(1)})a_{(2)}m_{(2)} \\
&= \sum_{(m)} 1 \otimes m_{(1)} \otimes am_{(2)} \\
&= a\psi(m); \\
\psi(xm) &= \sum_{(m)} 1 \otimes m_{(1)} \otimes m_{(2)}\langle m_{(3)}, x\rangle \\
&= \sum_{(m)} 1 \otimes m_{(1)} \otimes R_x(m_{(2)}) \\
&= x\psi(m).
\end{aligned}$$

This proves the first part of the proposition.

Now for any $x \in \mathrm{U}_q(\mathfrak{l})$ and $m \in M$, a routine calculation gives

$$\sum_{(x)}(x_{(1)} \otimes L_{x_{(2)}})\psi(m) = \sum_{(x),(m)} x_{(1)}(1 \otimes m_{(1)}) \otimes L_{x_{(2)}}(m_{(2)}) = \epsilon(x)\psi(m).$$

This implies that $Im\psi \subseteq \mathcal{S}(M_0)$ by recalling the definition of $\mathcal{S}(M_0)$ (cf. (3.2)).

To proceed further, we extend (3.2) slightly to allow for such Ξ that may be infinite direct sums of $\mathrm{U}_q(\mathfrak{l})$-modules in $\mathrm{U}_q(\mathfrak{l})$-**mod** so that we can define $\mathcal{S}(M)$ for the object M in $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$ under consideration here. We evidently have the $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module isomorphism $M_0 \cong M/IM$, and by Proposition 3.4 this is a $\mathrm{U}_q(\mathfrak{l})$-module isomorphism as well. Thus we have the short exact sequence $0 \longrightarrow IM \longrightarrow M \longrightarrow M_0 \longrightarrow 0$ of $\mathrm{U}_q(\mathfrak{l})$-modules. Tensor it with $\mathcal{A}(\mathfrak{g})$ over $\mathbb{C}$ and then take $\mathrm{U}_q(\mathfrak{l})$-invariants. Since all the $\mathrm{U}_q(\mathfrak{l})$-modules involved are semi-simple, we arrive at the following short exact sequence of $\mathrm{U}_q(\mathfrak{g})$-modules:

$$0 \longrightarrow \mathcal{S}(IM) \longrightarrow \mathcal{S}(M) \longrightarrow \mathcal{S}(M_0) \longrightarrow 0. \tag{3.9}$$

Denote by $\tilde{\psi}$ the surjective map in (3.9). It is given for all $\sum_i m_i \otimes_{\mathbb{C}} f_i \in \mathcal{S}(M)$ by

$$\tilde{\psi} : \sum_i m_i \otimes_{\mathbb{C}} f_i \mapsto \sum_i 1 \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} m_i \otimes_{\mathbb{C}} f_i.$$

One can see this very easily, e.g., by inspecting the formulae in Remark 3.6 below. Following [5], we consider the $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module morphism defined by the same formula (3.3) but for the object M in $\mathcal{M}(\mathfrak{g}, \mathfrak{l})$ under consideration here. Then equation (3.4) remains valid. An explicit calculation shows that for all $m \in M$ and $a \in \mathcal{A}(\mathfrak{g}, \mathfrak{l})$,

$$\psi(am) = \sum_{(m)} 1 \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} m_{(1)} \otimes_{\mathbb{C}} am_{(2)} = 1 \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} \kappa^{-1}(m \otimes_{\mathbb{C}} a).$$

In view of equation (3.4), this shows that $Im\psi = Im\tilde{\psi}$. Now it follows from the short exact sequence (3.9) that $Im\psi \supseteq \mathcal{S}(M_0)$, and this completes the proof of the lemma. □

Remark 3.6. If M is an object in $\mathcal{M}(\mathfrak{g},\mathfrak{l})$, we can describe $\mathcal{S}(M)$ and $\mathcal{S}(M_0)$ very explicitly. One can show that $\mathcal{S}(M)$ is spanned over $\mathbb{C}$ by the elements

$$\sum_{(m),(f)} m_{(1)} \otimes_{\mathbb{C}} f_{(2)} \int p(m_{(2)} S(f_{(1)})), \quad \forall m \in M, f \in \mathcal{A}(\mathfrak{g}),$$

where $p : \mathcal{A}(\mathfrak{g}) \longrightarrow \mathcal{A}(\mathfrak{l})$ is the Hopf algebra map defined in Section 2, and $\int$ is the invariant integral on $\mathcal{A}(\mathfrak{l})$. Also $\mathcal{S}(M_0)$ is spanned over $\mathbb{C}$ by the elements

$$\sum_{(m),(f)} 1 \otimes_{\mathcal{A}(\mathfrak{g},\mathfrak{l})} m_{(1)} \otimes_{\mathbb{C}} f_{(2)} \int p(m_{(2)} S(f_{(1)})), \quad \forall m \in M, f \in \mathcal{A}(\mathfrak{g}).$$

Lemma 3.7. *If M is a free $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module, then $M \cong \mathcal{S}(M_0)$.*

Proof. We claim that the desired isomorphism is provided by the map ψ (cf. (3.8)). To prove this, it suffices to show that ψ is injective in view of the previous lemma. Let $\{v^i\}_{i=1}^n$ be a basis of M. Let $m = \sum_{i=1}^n a^i v^i$ belong to the kernel of ψ, where $a^i \in \mathcal{A}(\mathfrak{g},\mathfrak{l})$. For any $x \in \mathrm{U}_q(\mathfrak{g})$, $a^i(k_j^{\pm 1} x) = a^i(x)$ for all j. Since $\mathrm{U}_q(\mathfrak{g})$ is spanned by monomials in the generators $e_j, f_j, k_j^{\pm 1}$, $j = 1, 2, \ldots, r$, with elements $k_j^{\pm 1}$ positioned on the left, a^i is equal to zero if it annihilates the monomials in e_j, f_j, $j = 1, 2, \ldots, r$. Now $\psi(m)(1) = \sum_{i=1}^n 1 \otimes a^i(1) v^i = 0$. Since $\{1 \otimes v^i\}_{i=1}^n$ forms a basis of M_0, we have $a^i(1) = 0$, $\forall i$.

For any generator e_j of $\mathrm{U}_q(\mathfrak{g})$, a simple calculation shows that

$$0 = \psi(m)(e_j) \quad = \quad k_j^{-1}(\sum a^i(e_j)(1 \otimes v^i)).$$

Thus $a^i(e_j) = 0, \forall i, j$. Similarly, one can see that $a^i(f_j) = 0, \forall i, j$.

Define the degrees of e_i and f_i to be 1, and denoted by $deg(x)$ the degree of a monomial x in the elements e_j and f_j. We use induction on the degree of the monomial x to show that every a^i satisfies $a^i(x) = 0$ for all x. We have already proved this for $deg(x) = 0$ and 1. We have $\Delta(x) = x \otimes K + \sum_{deg(x_{(1)}) < deg(x)} x_{(1)} \otimes x_{(2)}$ for any monomial x with $deg(x) > 1$, where K is a monomial in $k_i^{\pm 1}$. Now

$$\begin{aligned}
\psi(m)(x) = & \sum_{i,(v),(x)} 1 \otimes a^i_{(1)} v^i_{(1)} \langle a^i_{(2)}, x_{(1)} \rangle \langle v^i_{(2)}, x_{(2)} \rangle \\
& = \sum_{i,(v)} 1 \otimes v^i_{(1)} \langle a^i, x \rangle \langle v^i_{(2)}, K \rangle \\
& + \sum_{i,(v)} \sum_{deg(x_{(1)}) < deg(x)} 1 \otimes v^i_{(1)} \langle a^i, x_{(1)} \rangle \langle v^i_{(2)}, x_{(2)} \rangle \\
& = \sum_{i,(v)} 1 \otimes v^i_{(1)} \langle a^i, x \rangle \langle v^i_{(2)}, K \rangle \qquad \text{by induction hypothesis} \\
& = K \left(\sum_i 1 \otimes a^i(x) v^i \right) = 0.
\end{aligned}$$

Thus $a^i(x) = 0$ for all monomials x and hence $a^i = 0$. This proves the injectivity of ψ when M is free, thus completes the proof of the lemma. □

Proposition 3.8. *Let M be a finitely generated* $\mathrm{U}_q(\mathfrak{g})$*-equivariant projective* $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$*-module. Then* $M \cong \mathcal{S}(M_0)$.

Proof. Since M is a finitely generated $\mathrm{U}_q(\mathfrak{g})$-equivariant projective $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module, there exists another $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module N such that $M \oplus N = F$, where F is a free $\mathrm{U}_q(\mathfrak{g})$-equivariant $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-module of some finite rank. Let ψ be the map defined by equation (3.8) but for F. We get

$$\psi(M) \subseteq \mathcal{S}(M_0), \quad \psi(N) \subseteq \mathcal{S}(N_0).$$

By Lemma 3.7, $\psi(M \oplus N) = \psi(M) \oplus \psi(N) = \mathcal{S}(F_0) = \mathcal{S}(M_0) \oplus \mathcal{S}(N_0)$. Thus we must have $M \cong \mathcal{S}(M_0)$. □

Let $\mathcal{E}' : \mathcal{P}(\mathfrak{g}, \mathfrak{l}) \longrightarrow \mathrm{U}_q(\mathfrak{l})$-**mod** be the restriction to $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$ of the functor $\mathcal{E}$ defined by (3.7). Denote by $K(\mathrm{U}_q(\mathfrak{l})$-**mod**$)$ and $K(\mathcal{P}(\mathfrak{g}, \mathfrak{l}))$ respectively the Grothendieck groups of $\mathrm{U}_q(\mathfrak{l})$-**mod** and $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$. Then the functors $\mathcal{S}$ and $\mathcal{E}'$ induce homomorphisms

$$\mathcal{S}_* : K(\mathrm{U}_q(\mathfrak{l})\text{-}\mathbf{mod}) \longrightarrow K(\mathcal{P}(\mathfrak{g}, \mathfrak{l})), \quad \mathcal{E}'_* : K(\mathcal{P}(\mathfrak{g}, \mathfrak{l})) \longrightarrow K(\mathrm{U}_q(\mathfrak{l})\text{-}\mathbf{mod})$$

between the Grothendieck groups. Proposition 3.8 implies that $\mathcal{S}_*$ is surjective and $\mathcal{S}_* \circ \mathcal{E}'_*$ is the identity map on $K(\mathcal{P}(\mathfrak{g}, \mathfrak{l}))$. In fact we have the following result.

Theorem 3.9. *The Grothendieck groups of* $\mathrm{U}_q(\mathfrak{l})$-**mod** *and of* $\mathcal{P}(\mathfrak{g}, \mathfrak{l})$ *are isomorphic.*

Proof. In view of the preceding discussion on the maps $\mathcal{S}_*$ and $\mathcal{E}'_*$, it suffices to show either the injectivity of $\mathcal{S}_*$ or surjectivity of $\mathcal{E}'_*$. We shall prove the latter by showing that $\mathcal{S}(V)_0$ is isomorphic to V for any object V in $\mathrm{U}_q(\mathfrak{l})$-**mod**.

Since $\mathrm{U}_q(\mathfrak{l})$-**mod** is semi-simple, and the functor $\mathcal{S}$ is exact, we can assume that V is simple. Let W be the irreducible $\mathrm{U}_q(\mathfrak{g})$-module with highest weight λ such that the highest weight of V belongs to the Weyl group orbit of λ. Then the restriction of W contains exactly one $\mathrm{U}_q(\mathfrak{l})$-submodule isomorphic to V. Embed V in W, and let $V^\perp$ denote the $\mathrm{U}_q(\mathfrak{l})$-submodule of W such that $V \oplus V^\perp = W$. Then $\mathcal{S}(V)_0 \oplus \mathcal{S}(V^\perp)_0 = \mathcal{S}(W)_0 \cong W$. We have $\dim \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l})}(W, S(V)_0) \geq 1$ and the inequality will necessarily be strict if $\mathcal{S}(V)_0$ is not simple. Proposition 3.8 and Frobenius reciprocity [5] together lead to the vector space isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l})}(W, \mathcal{S}(V)_0) &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g})}(W, \mathcal{S}(\mathcal{S}(V)_0)) \\ &= \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g})}(W, \mathcal{S}(V)) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l})}(W, V) \\ &= \mathbb{C}. \end{aligned}$$

This shows that $\mathcal{S}(V)_0$ is simple.

Let $\eta : W \longrightarrow V$ be the projection onto V, and consider $\tilde{V} := (\eta \otimes \mathrm{id}_{\mathcal{A}(\mathfrak{g})})\delta(V)$, where δ is the right $\mathcal{A}(\mathfrak{g})$-comodule action on W. We can easily show by straightforward computations that $\tilde{V}$ is contained in $\mathcal{S}(V)$ and forms a $\mathrm{U}_q(\mathfrak{l})$-submodule isomorphic to V. If we can also show that $\tilde{V}_0 := (\mathcal{A}(\mathfrak{g}, \mathfrak{l})/I) \otimes_{\mathcal{A}(\mathfrak{g}, \mathfrak{l})} \tilde{V}$ is a non-zero subspace of $\mathcal{S}(V)_0$, then by Proposition 3.4 (or by direct inspection of the $\mathrm{U}_q(\mathfrak{l})$-action) it must be a $\mathrm{U}_q(\mathfrak{l})$-submodule of $\mathcal{S}(V)_0$ isomorphic to the irreducible module V. Since $\mathcal{S}(V)_0$ is irreducible, we must have $\mathcal{S}(V)_0 \cong V$.

To complete the proof, we need to show that $\tilde{V}_0 \neq 0$. Note that there is a well defined linear map $\mathcal{S}(V)_0 \longrightarrow V$ given by $1 \otimes \zeta \mapsto (\mathrm{id}_V \otimes \epsilon_0)\zeta$ for all $\zeta \in \mathcal{S}(V)$, where ϵ_0 is the co-unit of $\mathcal{A}(\mathfrak{g})$ defined by $\epsilon_0(f) = f(1)$ for all $f \in \mathcal{A}(\mathfrak{g})$. The image of $\tilde{V}_0$ under this map is nonzero (in fact is V).

This proves the surjectivity of $\mathcal{E}'_*$, thus completes the proof of the theorem. □

4. Discussions

The results of the previous sections may be placed in a natural framework, which we discuss here. Let us return to general properties of the category $\mathcal{P}(\mathfrak{g},\mathfrak{l})$ of equivariant quantum homogeneous vector bundles.

Proposition 4.1. *An object is projective in $\mathcal{M}(\mathfrak{g},\mathfrak{l})$ if and only if it belongs to $\mathcal{P}(\mathfrak{g},\mathfrak{l})$.*

Proof. The "only if part" is obvious. To prove the "if part", we note that if P is an object in $\mathcal{P}(\mathfrak{g},\mathfrak{l})$, then $P = \mathcal{S}(P_0)$ by Proposition 3.8. We can always find a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module W_0 such that its restriction to a $\mathrm{U}_q(\mathfrak{l})$-module decomposes into $W_0 = P_0 \oplus Q_0$. Let $Q = \mathcal{S}(Q_0)$, then by [5], $P \oplus Q \cong W_0 \otimes \mathcal{A}(\mathfrak{g},\mathfrak{l})$ in $\mathcal{M}(\mathfrak{g},\mathfrak{l})$. We recall that $\mathrm{U}_q(\mathfrak{g})$ acts on the right hand side through the opposite co-multiplication Δ'. Therefore, $W_0 \otimes 1$ forms a $\mathrm{U}_q(\mathfrak{g})$-submodule of $W_0 \otimes \mathcal{A}(\mathfrak{g},\mathfrak{l})$ isomorphic to W_0 itself.

In order to prove that P is projective in $\mathcal{M}(\mathfrak{g},\mathfrak{l})$, it suffices to show that $W_0 \otimes \mathcal{A}(\mathfrak{g},\mathfrak{l})$ is. Given morphisms $f : M \longrightarrow N$ and $\beta : W_0 \otimes \mathcal{A}(\mathfrak{g},\mathfrak{l}) \longrightarrow N$ in $\mathcal{M}(\mathfrak{g},\mathfrak{l})$ with f being surjective, we always have an $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-map $i : W_0 \otimes \mathcal{A}(\mathfrak{g},\mathfrak{l}) \longrightarrow M$ such that $f \circ i = \beta$. As a $\mathrm{U}_q(\mathfrak{g})$-module, M decomposes into the direct sum $M = \ker f \oplus M_1$. Let π denote the projection on to M_1. Define the following $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-map

$$\tilde{i} : W_0 \otimes \mathcal{A}(\mathfrak{g},\mathfrak{l}) \longrightarrow M, \qquad w \otimes a \mapsto a(\pi \circ i(w \otimes 1)). \tag{4.1}$$

Then $f \circ \tilde{i} = \beta$. The proposition will follow if we can show that $\tilde{i}$ is also a $\mathrm{U}_q(\mathfrak{g})$-map.

For any $w \in W_0$ and $x \in \mathrm{U}_q(\mathfrak{g})$, we let $d(w,x) := x\tilde{i}(w \otimes 1) - \tilde{i}(x(w \otimes 1))$, which belongs to M_1 since $W_0 \otimes 1$ forms a $\mathrm{U}_q(\mathfrak{g})$-submodule of $W_0 \otimes \mathcal{A}(\mathfrak{g},\mathfrak{l})$. On the other hand,

$$f(d(w,x)) = x\beta(w \otimes 1) - \beta(x(w \otimes 1)) = 0,$$

that is, $d(w,x) \in \ker f$. Hence $d(w,x) = 0$. It follows that

$$x\tilde{i}(w \otimes a) - \tilde{i}(x(w \otimes a)) = \sum_{(x)} R_{x_{(1)}}(a) d(w, x_{(2)}) = 0$$

for all $a \in \mathcal{A}(\mathfrak{g},\mathfrak{l})$. Therefore, $\tilde{i}$ is indeed a $\mathrm{U}_q(\mathfrak{g})$-map, and this completes the proof. □

It immediately follows from Proposition 4.1 that $\mathcal{P}(\mathfrak{g},\mathfrak{l})$ is an exact category. Therefore, one may follow Quillen's construction to define a "quantum group equivariant algebraic K-theory" for the algebra $\mathcal{A}(\mathfrak{g},\mathfrak{l})$. Let $\mathcal{Q}(\mathfrak{g},\mathfrak{l}) = Q\mathcal{P}(\mathfrak{g},\mathfrak{l})$ be the Quillen category of the exact category $\mathcal{P}(\mathfrak{g},\mathfrak{l})$ of finitely generated $\mathrm{U}_q(\mathfrak{g})$-equivariant projective $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-modules. Denote by $B\mathcal{Q}(\mathfrak{g},\mathfrak{l})$ the classifying space of the category $\mathcal{Q}(\mathfrak{g},\mathfrak{l})$ (see [11, Sections 3, 4] for the definitions of the classifying space of a category and the Quillen category of an exact category). Then the $\mathrm{U}_q(\mathfrak{g})$-equivariant K-groups $K_i^{\mathrm{U}_q(\mathfrak{g})}(\mathcal{A}(\mathfrak{g},\mathfrak{l}))$ of $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ may be defined as the homotopy groups $\pi_{i+1}(B\mathcal{Q}(\mathfrak{g},\mathfrak{l}))$, $i \geq 0$, of $B\mathcal{Q}(\mathfrak{g},\mathfrak{l})$. It is a standard fact that the K_0-group $\pi_1(B\mathcal{Q}(\mathfrak{g},\mathfrak{l}))$ is the Grothendieck group of $\mathcal{P}(\mathfrak{g},\mathfrak{l})$. This provides a natural framework for understanding Theorem 3.9.

It will be very interesting to systematically develop this K-theory.

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