# Geometry & Groups

Stephan Tillmann

*Version 12.1.2012*

## 1 Möbius transformations and inversions

These notes give an approximation of the first lecture. The pictures are not reproduced here, but most of them are contained in the handout "Classification of Möbius transformations."

### 1.1 Linear and anti-linear fractional maps

Denote $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere, and define

$$M(z) = \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{C}$ satisfy $ad - bc \neq 0$, and

$$M(-\frac{d}{c}) = \infty \qquad \text{and} \qquad M(\infty) = \frac{a}{c}.$$

Then $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a continuous function, called a *Möbius transformation* or *linear fractional map*. I talked about the continuity, and the topology of the Riemann sphere via the stereographic projection. I also highlighted some properties of stereographic projection.

We now look at the action of some simple Möbius transformation.

(1) The map $z \to z+b,\ b \neq 0$, fixes $\infty$ and acts on $\mathbb{C}$ as a translation by $b$.

(2) The map $z \to az,\ a \neq 1$, fixes $0$ and $\infty$, hence again acts on $\mathbb{C}$:
   - (a) if $a = e^{i\vartheta} \in S^1,\ \vartheta \notin 2\pi\mathbb{Z}$, this is a rotation by $\vartheta$ about $0$;
   - (b) if $a \in \mathbb{R}_+ \setminus \{1\}$, then this is a dilation by $a$;
   - (c) if $a \in \mathbb{C} \setminus (S^1 \cup \mathbb{R}_+)$ this is a composition of rotation and dilation.

(3) The map $z \to \frac{1}{z}$ is called complex (standard) inversion. This interchanges $0$ and $\infty$. One can view it as a composition of geometric inversion in the unit circle, $z \to \frac{1}{\bar{z}}$, and reflection in the $x$–axis, $z \to \bar{z}$.

The inversion leads us to also consider *anti-linear fractional maps*, which are the maps of the form:

$$M(z) = \frac{a\bar{z}+b}{c\bar{z}+d},$$

where $a,b,c,d \in \mathbb{C}$ satisfy $ad - bc \neq 0$.

Key features of the complex inversion are: It only fixes $-1$ and $1$, it takes the unit circle to itself and interchanges the two complementary regions.

Key features of geometric inversion are: It fixes the whole unit circle, and interchanges the two complementary regions. Moreover, it preserves circles orthogonal to the unit circle. We then discussed (in this lecture and the next) a method to determine the image of a point by ruler and compass constructions.

We next pictured the action of the above transformations on the plane and the Riemann sphere, referring to the pictures from Tristan Needham's *Visual Complex Analysis*, giving the names parabolic (1), elliptic (2a), hyperbolic (2b) and loxodromic (2c). I claimed that every Möbius transformation is of this form, and we identified complex inversion as a rotation by $\pi$, hence an elliptic transformation.

**Exercise 1.1** *Let* $M(z) = \dfrac{az+b}{cz+d}$, *where* $a,b,c,d \in \mathbb{C}$ *satisfy* $ad-bc \neq 0$.

(1) *$M$ is bijective.*

(2) $\mathrm{Fix}(M) = \{z \in \widehat{\mathbb{C}} \colon M(z) = z\}$ *is either all of* $\widehat{\mathbb{C}}$ *(and hence $M$ is the identity map), or it consists of precisely one point or of precisely two points.*

(3) *If* $N(z) = \dfrac{a'z+b'}{c'z+d'}$, *then $M = N$ if and only if there is $w \in \mathbb{C}$ such that* $a' = wa,\ b' = wb,\ c' = wc,\ d' = wd$.

## 1.2 Algebraic structure

I left it as an exercise to show that the set of all Möbius transformations forms a group, which I denote $\mathrm{Möb}(\widehat{\mathbb{C}})$. With the expression

$$M(z) = \frac{az+b}{cz+d},$$

we can associate the matrix

$$[M] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $[M] \in GL_2(\mathbb{C})$, but due to the above exercise, the assignment $M \to [M]$ is not well-defined. However, the map $GL_2(\mathbb{C}) \to \mathrm{Möb}(\widehat{\mathbb{C}})$ defined by sending the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the Möbius transformation defined by

$$z \mapsto A \cdot z = \frac{az+b}{cz+d}$$

is not only well-defined, but it is a surjective group homomorphism. Here one needs to check that multiplication of matrices in $GL_2(\mathbb{C})$ corresponds to composition of Möbius transformations.

Using Exercise 1.1, one sees that the kernel of this homomorphism is precisely the subgroup:

$$\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C}^*\} = \{aE \mid a \in \mathbb{C}^*\} \cong \mathbb{C}^*.$$

The matrices of the form $aE$ are called *scalar matrices* and form the centre of $GL_2(\mathbb{C})$ (i.e. they are precisely the matrices that commute with every other matrix). By the first isomorphism theorem, this gives:

$$GL_2(\mathbb{C})/\{aE \mid a \in \mathbb{C}^*\} \cong \mathrm{Möb}(\widehat{\mathbb{C}}).$$

Now elements of $GL_2(\mathbb{C})$ have eigenvectors and eigenvalues. We will see that an eigenvector of an element of $GL_2(\mathbb{C})$ corresponds to a fixed point of the associated Möbius transformation.

### 1.3 Homogeneous coordinates, eigenvectors and fixed points

A given $z \in \mathbb{C}$ can be written as a fraction $z = \frac{z_1}{z_2}$, where $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}^*$. The expression is not unique since

$$z = \frac{z_1}{z_2} = \frac{wz_1}{wz_2}$$

for all $w \in \mathbb{C}^*$. There is a natural equivalence relation on $\mathbb{C}^2 \setminus \{(0,0)\}$, by declaring

$$(z_1, z_2) \sim (z_1', z_2')$$

if and only if there is $w \in \mathbb{C}^*$ such that $z_1' = wz_1$ and $z_2' = wz_2$. I'll denote the equivalence class of $(z_1, z_2)$ by $[z_1, z_2]$. The set of equivalence classes

$$\mathbb{C}P^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$$

is called the *complex projective line.*

Note that $z = \frac{z_1}{z_2} \to [z_1, z_2]$ gives a well-defined injective map $\mathbb{C} \to \mathbb{C}P^1$, whose image consists precisely of the equivalence classes $[z_1, z_2]$, where $z_2 \neq 0$. We therefore map $\infty$ to the unique equivalence class that is not in the image of $\mathbb{C}$, namely: $\infty \to [z_1, 0]$, where $z_1 \in \mathbb{C}^*$. I'll mention an irrelevant, but nice, fact: the complex projective line $\mathbb{C}P^1$ inherits a quotient topology from $\mathbb{C}^2 \setminus \{(0,0)\}$, and the map $\widehat{\mathbb{C}} \to \mathbb{C}P^1$ is a homeomorphism.

For $z \in \widehat{\mathbb{C}}$, we call its image $[z_1, z_2]$ in $\mathbb{C}P^1$ its *homogeneous coordinate*.

If $A \in GL_2(\mathbb{C})$ and $B = wA$, then $v \in \mathbb{C}^2$ is an eigenvector of $A$ if and only if it is an eigenvector of $B$. So the following fact may now not seem so surprising.

**Exercise 1.2** *$M(z) = z$ if and only if $(z_1, z_2)$ is an eigenvector of $[M]$, where $[z_1, z_2]$ is the homogeneous coordinate of $z$ and $[M]$ is any matrix representing $M$.*

A natural question is whether eigenvalues also carry geometric information. At first glance, they should not since if $B = wA$ and $Av = \lambda v$, then $Bv = (\lambda w)v$, which implies that for every $\lambda \in \mathbb{C}^*$ and every Möbius transformation $M$, there is a matrix representing $M$ with eigenvalue $\lambda$. The answer lies in a natural *normalisation*.

### 1.4 Normalisation, eigenvalues and a characterisation using the trace

Consider the subgroup $SL_2(\mathbb{C}) \leq GL_2(\mathbb{C})$ consisting of all matrices with determinant one. Then there is an epimorphism

$$SL_2(\mathbb{C}) \twoheadrightarrow \text{Möb}(\widehat{\mathbb{C}}),$$

and the kernel consists only of $\{\pm E\}$, giving

$$PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm E\} \cong \text{Möb}(\widehat{\mathbb{C}}).$$

Now the eigenvalues $\lambda, \mu$ of $A \in SL_2(\mathbb{C})$ satisfy $\lambda\mu = \det(A) = 1$, so $\mu = \lambda^{-1}$. We then have

$$\lambda + \lambda^{-1} = \text{tr}(A),$$

where the trace $\text{tr}(A)$ is the sum of the diagonal entries. You can either appeal to the Jordan Normal Form or do an elementary calculation to show that any $A \in SL_2(\mathbb{C})$ can be conjugated by an element $P \in SL_2(\mathbb{C})$ to give a matrix $PAP^{-1}$ which is of precisely one of the following forms:

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ with } \lambda^2 \neq 1.$$

The first corresponds to the identity $z \to z$, the second to the translation $z \to z+1$ and the last to the transformation $z \to \lambda^2 z$, which is either elliptic, hyperbolic or loxodromic. Writing $\lambda^2 = re^{i\vartheta}$, I left it as an exercise to check that one obtains the following characterisation using the trace:

$$\begin{array}{lcl}
z \mapsto A \cdot z \text{ is elliptic} & \Longleftrightarrow & \operatorname{tr} A \in \mathbb{R} \text{ and } |\operatorname{tr} A| < 2; \\
z \mapsto A \cdot z \text{ is parabolic} & \Longleftrightarrow & \operatorname{tr} A = \pm 2 \text{ and } A \neq \pm E; \\
z \mapsto A \cdot z \text{ is hyperbolic} & \Longleftrightarrow & \operatorname{tr} A \in \mathbb{R} \text{ and } |\operatorname{tr} A| > 2; \\
z \mapsto A \cdot z \text{ is loxodromic} & \Longleftrightarrow & \operatorname{tr} A \notin \mathbb{R}.
\end{array}$$

We will get more geometry from the trace in week 3!

### 1.5 Three final comments

The comments consisted of two exercises and a sneak preview:

**Exercise 1.3** *A Möbius transformation is uniquely determined by its action on any three pairwise distinct points in $\widehat{\mathbb{C}}$. Moreover, for any two triples of pairwise distinct points, say $p_1, p_2, p_3$ and $q_1, q_2, q_3$ there is a (unique) Möbius transformation taking $p_k \to q_k$ for each $k \in \{1,2,3\}$.*

**Exercise 1.4** *Define the* cross ratio *of pairwise distinct points $a, b, c, d \in \widehat{\mathbb{C}}$ (in this order!) by:*

$$CR[a,b,c,d] = \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

*If one of the points is $\infty$, then the correct expression is obtained by continuous extension. For instance, if $a = \infty$, consider the limiting value as $a \to \infty$, giving $CR[\infty,b,c,d] = \dfrac{(b-d)}{(b-c)}$.*

*Then for any Möbius transformation $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, we have*

$$CR[M(a), M(b), M(c), M(d)] = CR[a,b,c,d].$$

*Moreover, the $4! = 24$ permutations of $a, b, c, d$ give rise to (generically) 6 distinct cross ratios.*

**Sneak preview** The group $\text{Möb}(\widehat{\mathbb{C}})$ contains the *orientation preserving* isometries of the euclidean plane, the round 2–sphere and the hyperbolic plane. (The orientation reversing isometries of these geometries correspond to certain anti-linear fractional transformations). I denoted these three groups of orientation preserving isometries respectively by:

$$\operatorname{Isom}^+(\mathbf{E}^2), \operatorname{Isom}^+(\mathbb{S}^2) \text{ and } \operatorname{Isom}^+(\mathbf{H}^2).$$

The key to this lies in the fact that $\text{Möb}(\widehat{\mathbb{C}}) \cong \operatorname{Isom}^+(\mathbf{H}^3)$, which we'll talk about in week 3.