Topology and Geometry of Three-Dimensional Manifolds

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Ja mach nur einen Plan,	[Make yourself a plan,]
sei nur ein grosses Licht!	[be a clever chap!]
Und mach dann noch ’nen zweiten Plan,	[And then make another plan,]
gehn tun sie beide nicht.	[neither of them works.]
Das Lied von der Unzulänglichkeit	[The song of the insufficiency]
menschlichen Strebens	[of human endeavour]
Bertolt Brecht	

Synopsis

The main topics of this course are cornerstones of 3–dimensional topology. Due to Thurston’s Geometrisation Conjecture (proven by Perelman), every compact 3–manifold can be cut along surfaces into pieces that admit geometric structures. This decomposition is unique: it consists of the prime decomposition of Kneser and Milnor and the characteristic submanifold decomposition of Jaco, Shalen and (independently) Johannson. Both of these decomposition theorems will be proved in this course. Modern proofs of the foundational results, the disc theorem of Papakyriakopoulos (also known as *Dehn’s lemma and the loop theorem*) and his sphere theorem, will be given. Many problems about 3–manifolds are solved using the concepts of incompressible surfaces and hierarchies due to Haken, and practical algorithms for decision problems use the theory of normal surfaces. The course will end with a *weak geometrisation theorem* for Haken 3–manifolds—focussing not on the geometry of the manifolds, but rather one the geometry of their fundamental groups.

This course is based on the following sources (full details are given in the bibliography):

[1] Marc Lackenby: *Three-Dimensional Manifolds*

[2] Alan Hatcher: *Notes on Basic 3–Manifold Topology*

[3] Iain Aitchinson and Hyam Rubinstein: *Localising Dehn’s lemma and the loop theorem in 3–manifolds*

[4] Bus Jaco and Hyam Rubinstein: *PL Equivariant Surgery and Invariant Decompositions of 3–Manifolds*

[5] Walter Neumann: *Notes on geometry and 3-manifolds*

Prerequisites

Basic general topology (eg. compactness, quotient topology)

Basic algebraic topology (eg. homotopy, fundamental group, homology, covering spaces)

If you have taken *MATH3961 Metric Spaces* and *PMH1 Algebraic Topology*, then you know more than you need for this course. A convenient place to revise the assumed knowledge is Armstrong’s *Basic topology* [7].

Lectures

Lectures are given on Mondays, 13:00–15:00 in Carslaw 830, except for Week 6, when a practice session will be held by Robert Haraway. A number of exercises will be given in each lecture, and solutions can be handed in for assessment (see below). I'm planning to take a break in the middle of each lecture for discussion of problems, and students will be invited to present their solutions in class. As there are no formal tutorials, you should get together with other students to discuss the course contents and attempt additional problems from the references.

Assessment

(1) Exercises: 3% (at most) may be earned every week (starting in the second week) in which you turn in a single worked exercise at the beginning of the lecture. Please limit your solution to one piece of paper – if you find you need more space, write out a complete solution and then rewrite with conciseness in mind. Note that $12 \times 3\% = 36\%$.

(2) One scribe assignment, worth 10%. Each lecture will have a designated scribe. The scribe's job is to faithfully take notes for that lecture, write them up (preferably in latex, but this is not a requirement), and send them to the lecturer. Pictures can be drawn by hand and scanned. Make sure to include everything that was covered in the lecture. These notes will be posted here, and serve as an informal reference for the course.

(3) A written exam, worth 60%, to be held in the examination period at the end of Semester 2, covering the whole content of the course.

Note that $36+10+60 = 106$.

Outline

Below is a *tentative* overview of the lectures, together with some references, followed by a week-by-week summary. Each lecture is held in a 2 hour time-slot, with a break in the middle for relaxation and discussion of problems.

The rough plan is to first cover standard background material following §§1–4, 6 of Lackenby [1] in the first four lectures. Extra notes will be provided for lectures 5 and 6—the former will be based on but substantially deviate from [1, §11], and the latter is similarly related to Hatcher [2, §1.1].

Lectures 7–9 give the simplified proofs due to Aitchison, Jaco and Rubinstein [3, 4] of the foundational topological results of Papakyriakopoulos—Dehn's lemma, the loop theorem and the sphere theorem. Lecture 10 provides the topological foundation for the *geometric* decomposition of a 3–manifold, following the approach of Neumann and Swarup [5, **?**]. Lecture 11 gives a very brief introduction to Thurston's geometries for 3–manifolds. Lecture 12 proves a weak version of the hyperbolisation theorem for atoroidal Haken 3–manifolds by showing that each of these manifold has negatively curved fundamental group in the sense of Gromov.

In week 6, there will be a practice class to highlight algorithms for decision problems about 3–manifolds. A practice sheet with various tasks will be provided.

Week 1: Manifolds

Definition of manifold, Examples (gluings, knots, mapping cylinder), Connected sum
The classification problem, TOP, PL and DIFF

References: [1, §1]

Week 2: Elements of PL topology

Piecewise linar maps, Submanifolds, Isotopy
Regular neighbourhoods, Handle structures, General position

References: [1, §2 & §4] — Optional reading: [1, §5]

Week 3: Incompressible surfaces

Incompressible surfaces, Alexander's theorem

Week 4: Bundles and neighbourhoods

Regular neighbourhood, Fibre bundle, Tubular neighbourhood

References: [1, §6] — Optional reading: [1, §7]

Week 5: Normal surfaces

Normal surface theory, Normalisation of surfaces in triangulated 3–manifolds

References: [6], [1, §11], [2, §1.1]

Week 6: Regina (Practice class)

Computing with the normal surface calculator

References: [13] — Optional reading: Joel Hass, Joel, Jeffrey Lagarias and Nicholas Pippenger: *The computational complexity of knot and link problems*, J. ACM 46 (1999), no. 2, 185–211.

Week 7: Prime decomposition

Existence via Kneser-Haken finiteness, Matveev's uniqueness proof

References: [6], [2, §1.1]

Week 8: Hierarchies

Geometric hierarchies, Very short hierarchies

References: [3, §§2–3] — Optional reading: [1, §8, §10]

Week 9: Dehn's lemma and the loop theorem

Proof and Applications

References: [3, §4], [2, §3.1] — Optional reading: [1, §9]

Week 10: The sphere theorem

PL minimal surfaces, Equivariant sphere theorem, Classical sphere theorem

References: [4, §§2-4]

Week 11: Characteristic torus decomposition

JSJ decomposition, Seifert fibred manifolds

References: [5, §2], [2, §1.2, §2.1]

Week 12: Geometries for 3-manifolds

Geometric structures on manifolds, Developing map and holonomy, The eight Thurston geometries

References: [6] — Optional reading: [5, §1]

Week 13: Haken manifolds

Large scale geometry of groups and spaces (after Gromov), Word hyperbolic groups
Atoroidal Haken 3-manifold has word hyperbolic fundamantal group

References: [6] — Optional reading: [1, §12]

Bibliography

For the most part, the lectures and the primary sources should be more than sufficient. A number of references, which either offer a different approach or more detail, have been put on reserve in the library.

Primary sources

[1] Marc Lackenby: *Three-Dimensional Manifolds*, Lecture Notes for Graduate Course, Michaelmas 1999. Available from: `http://people.maths.ox.ac.uk/lackenby/`

[2] Alan Hatcher: *Notes on Basic 3–Manifold Topology* Available from `https://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html`

[3] Iain Aitchinson and Hyam Rubinstein: *Localising Dehn's lemma and the loop theorem in 3–manifolds*, Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 2, 281–292.

[4] William Jaco and Hyam Rubinstein: *PL Equivariant Surgery and Invariant Decompositions of 3–Manifolds*, Adv. in Math. 73 (1989), no. 2, 149–191.

[5] Walter D. Neumann: *Notes on geometry and 3-manifolds*. With appendices by Paul Norbury. Bolyai Soc. Math. Stud., 8, Low dimensional topology (Eger, 1996/Budapest, 1998), 191–267, János Bolyai Math. Soc., Budapest, 1999. Available from: `www.math.columbia.edu/~neumann/preprints/`

[6] Stephan Tillmann: *Notes to be provided*

Other main references on reserve in the library

[7] Mark Anthony Armstrong: *Basic topology*. Corrected reprint of the 1979 original. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1983.

[8] Gerhard Burde and Heiner Zieschang: *Knots*. Second edition. de Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 2003. (Available from the library web-site as e-book.)

[9] John Hempel: *3-manifolds*. Reprint of the 1976 original. AMS Chelsea Publishing, Providence, RI, 2004.

[10] Sergei Matveev: *Algorithmic topology and classification of 3-manifolds.* Algorithms and Computation in Mathematics, 9. Springer, Berlin, 2007. (Available from the library web-site as e-book.)

[11] Colin Patrick Rourke and Brian Joseph Sanderson: *Introduction to piecewise-linear topology.* Reprint. Springer Study Edition. Springer-Verlag, Berlin-New York, 1982.

[12] William P. Thurston: *Three-Dimensional Geometry and Topology*. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

Software

[13] Benjamin A. Burton and Ryan Budney and William Pettersson and others: *Regina: Software for 3-manifold topology and normal surface theory*, `http://regina.sourceforge.net/`, 1999–2014.

[14] Marc Culler, Nathan M. Dunfield, Matthias Goerner and Jeffrey R. Weeks: *SnapPy, a computer program for studying the geometry and topology of* 3*-manifolds*, `http://snappy.computop.org`.