Topology and Geometry of Three Dimensional Manifolds Lecture 5

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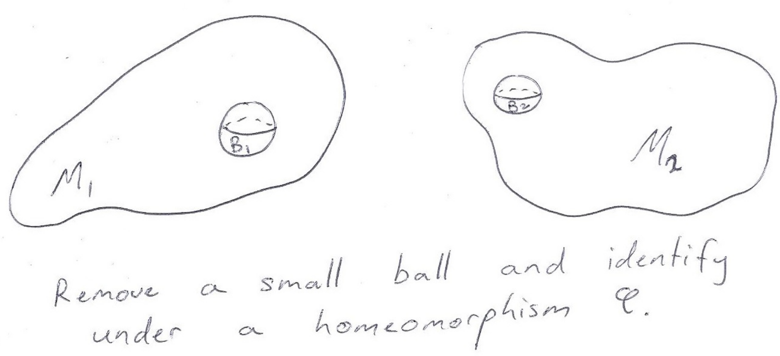
Last week we talked about Alexander's theorem. In particular, we had by Schöenflies-Alexander's theorem (Theorem 5 - Week 4) that every embedded 2-sphere in S^3 or $\mathbb{R}^3$ bounds a ball ($S^2 \hookrightarrow S^3$).

The above motivates the following definition:

Definition 1 (Irreducible 3-Manifold): The PL 3-manifold M^3 is irreducible if every PL-embedded 2-sphere in M^3 bounds a ball.

Remark 1: We will (and in fact, can) assume that M^3 is PL throughout. In fact, Moise [1952] showed that every topological 3-manifold has an essentially unique PL structure.

Definition 2 (Prime 3-Manifold): Let M_1 and M_2 be oriented 3-manifolds. Furthermore, suppose M^3 is prime such that $M^3 \cong M_1 \# M_2$. Then, $M_1 \cong S^3$ or $M_2 \cong S^3$.



$$M_1 \# M_2 = (M_1 \backslash B_1) \cup (M_2 \backslash B_2) / (\varphi : \partial B_1 \to \partial B_2)$$

where φ is orientation reversing. Note that it doesn't matter which φ we pick because all φ are isotopic.

Remark 2: If M^3 is a 3-manifold then

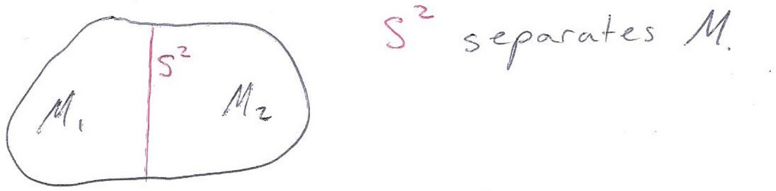
- $M^3 \# S^3 = M^3$
- M^3 irreducible $\Rightarrow M^3$ prime
- M^3 orientable, connected, irreducible and $M^3 \ncong B^3$ imples no component of ∂M^3 is homeomorphic to S^2.

Lemma 1: M connected, orientable and prime $\Rightarrow M$ is irreducible OR $M \cong S^2 \times S^1$

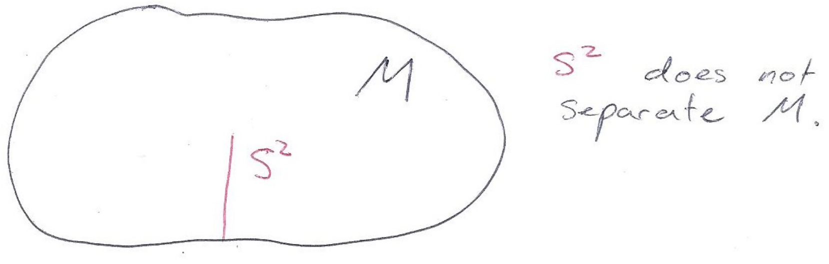
Proof. If every 2-sphere S^2 in M bounds a ball then M is irreducible.

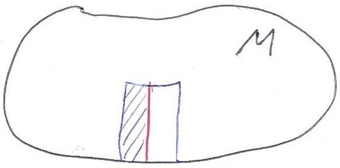
Otherwise, we can assume there is $S^2 \subseteq M$ that does not bound a ball. This gives us 2 cases:

Case 1: S^2 is separating M into two components M_1 and M_2 such that M_1 is not a ball and M_2 is not a ball (i.e. $M_1 \ncong B^3 \ncong M_2$). So $M = \overline{M_1} \# \overline{M_2}$ is not prime, where $\overline{M_k} = M_k \cup B^3$ for $k \in \{1, 2\}$.



Case 2: S^2 is non-separating. That is, $M \backslash S^2$ is connected.

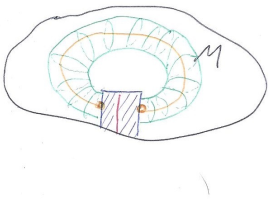




The product structure is an embedding

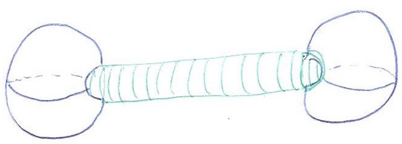
$h: S^2 \times [-1, 1] \longrightarrow M$

with $h(S^2 \times 0) = S^2$



Connect each $h(S^2 \times -1)$ and $h(S^2 \times 1)$ with a path in $M \setminus S^2$ meeting the spheres only in its endpoints. Then, fatten this path into a tube so that each of $h(S^2 \times -1)$ and $h(S^2 \times 1)$ is connected with a tube.

As the above diagrams suggest, in M, we have a neighbourhood of S^2 of the form $N = S^2 \times [0, 1]$ and an arc α in $\overline{M \setminus N}$ with the point in $\partial\alpha$ on $S^2 \times \{0\}$ and $S^2 \times \{1\}$.



So now this guy separates my manifold.

Question: what is this guy?

Answer: A 2-sphere.

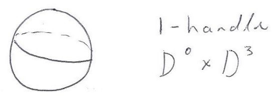
Hence, the sphere separates my manifold.

Consider $M' = (S^2 \times [0,1]) \cup (N(\alpha))$. Then $\partial M' = S^2$ (that is, $\partial M'$ is a 2-sphere).

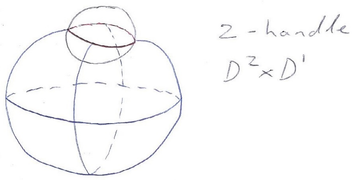
Now, as $\partial M'$ separates the prime manifold M it follows that at least one side of the separated prime manifold M must bound a ball. But $M' \ncong B^3$ since it contains a non-separable 2-sphere. Hence $\overline{M \backslash M'} \cong B^3$ and hence $M \cong S^2 \times S^1$. $\square$

One way to see the above lemma in action is to use handle decompositions. So, how do we build $S^2 \times S^1$?

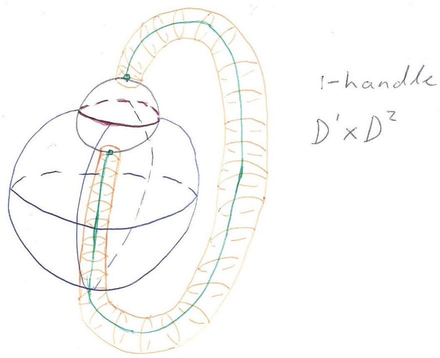
Take a 0-handle



Attach a 2-handle to the equator.



Attach a 1-handle disjointly.



Notice that we can reverse the order in which we attach these handles as the above handles are disjoint when they attach.

Now, the boundary of M' is a 2-sphere so we lastly attach a 3-handle over the top of our above handle decomposition.

Next on our agenda we will show a theorem by Kneser and Milnor. The existence part of the above Theorem was proved by Kneser [1929] and the uniqueness part was proved by Milnor [1961].

Theorem 1 [Kneser (1929) and Milnor (1961)]: If M^3 is a non-trivial orientable, connected and closed 3-manifold then M^3 is diffeomorphic to a finite connected sum of prime manifolds

$$M^3 = M_1 \# M_2 \# \cdots \# M_k$$

That is, M_1,...,M_k are prime and the above expression unique up to reordering.

Exercise 1: Suppose M is an orientable, compact and connected 3-manifold. Furthermore, suppose $S \subseteq M$ is a non-separating 2-sphere. Show that these assumptions imply $M \cong N\#(S^2 \times S^1)$ for some N. Deduce from this that there is a bound on the number if $S^2 \times S^1$ summands in M.

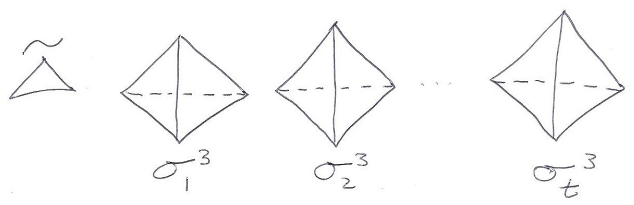
Hint: Kryptic hint for boundedness in Exercise 1

Exercise 2: Suppose $M^3 \subseteq S^3$ where M^3 is a compact 3-manifold and $|\partial M| = 1$ (i.e. there is only one boundary component). From these assumptions show that M is irreducible.

By Moise [1952], we know we can triangulate. Hence, for a 3-manifold, we set up a finite tirangulation (possible singular - i.e. not simplicial)

$$\tilde{\Delta} = \sigma_1^3 \cup \cdots \cup \sigma_t^3$$

where each σ_i^3 ($i \in \{1, 2, ..., t\}$) is a 3-simplex.



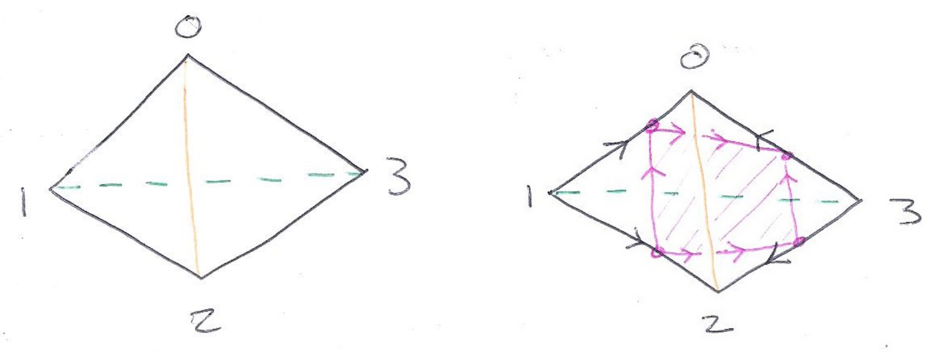
Next, we let ϕ be a collection of face pairings (i.e. a collection of linear homeomorphisms between faces). Note that any face pairing is uniquely determined by a bijection between vertices of faces. Hence,

$$M^3 = \tilde{\Delta}/\phi$$

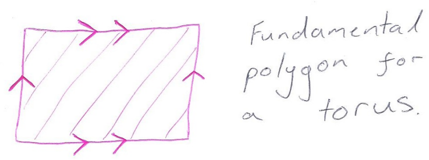
and so we have the natural quotient map

$$\rho : \tilde{\Delta} \to \tilde{\Delta}/\phi = M^3$$

Example 1: We want to identify [0,1,2] with [0,3,2]. We know $M = \sigma/\phi$. Here $\phi = \{[0, 1, 2] \leftrightarrow [0, 3, 2], [0, 1, 3] \leftrightarrow [2, 1, 3]\}$. Naturally, we want to know whether or not M is a manifold, and if so, which one is it?

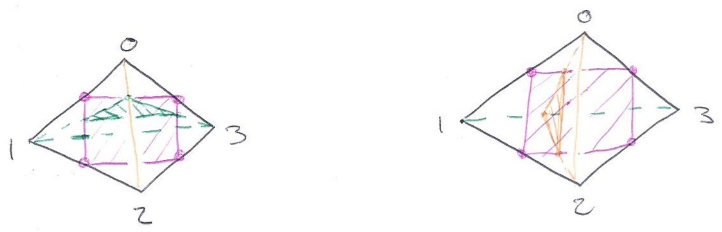


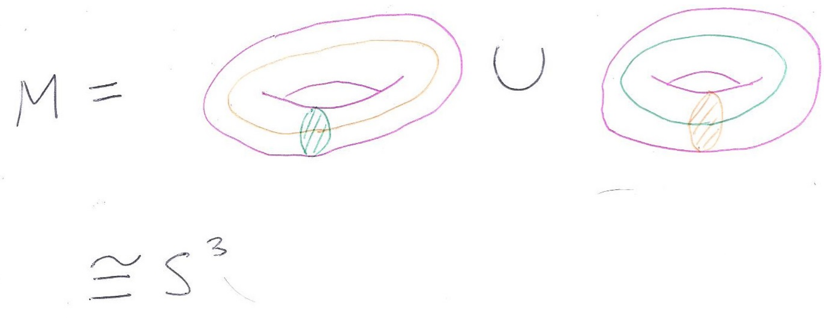
Notice that the torus can be found under idenitification of the edges.



Moreover, this torus is boundary of a neighbourhood of each the **orange** edge and the **green** edge. That is, the **pink** square is an embedded torus.

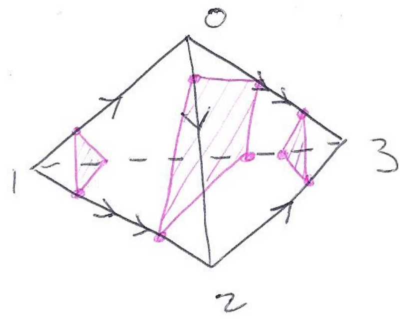
Next, we look at the cross sections produced



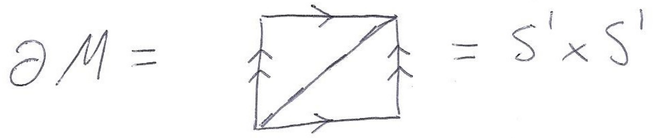


Using the above diagrams, we now know that M is a 3-manifold because we are gluing the above two tori together.

Example 2: Identifying with a twist.

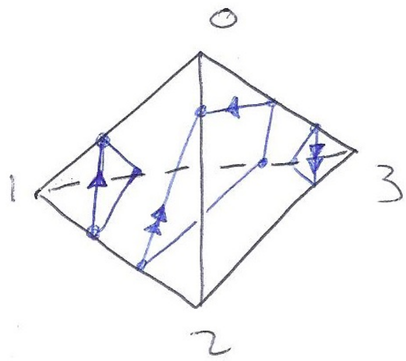


We know $M = \sigma/\phi$ where $\phi = \{[0, 1, 2] \leftrightarrow [2, 0, 3]\}$. Observe



Claim: M is a solid torus (i.e. $M \cong D^2 \times S^1$)

Claim: The three discs in σ map to an embedded disc D in M and $\overline{M \ D} \cong B^3$.

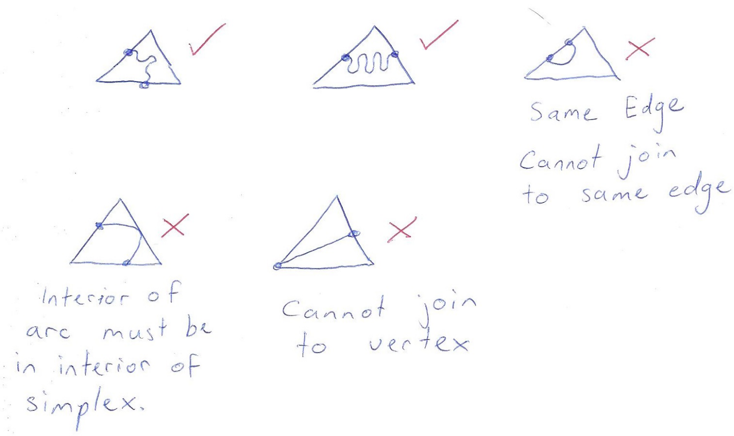


Definition 3 (Normal Arc): A properly embedded arc $(\alpha, \partial\alpha) \subset (\Delta^2, \partial\Delta^2)$ is normal if $\partial\alpha$ is on distinct edges of Δ^2 and $\alpha \cap \Delta^{(0)} = \emptyset$.

Remark 3: In general:

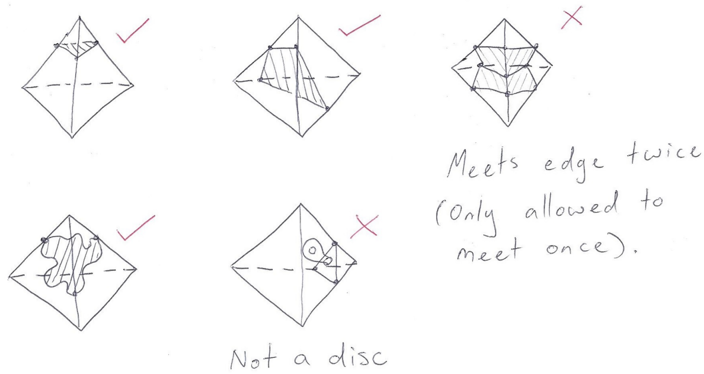
- Δ^k =k-simplex
- $\Delta^{(j)}$ =j-skeleton of the simplex Δ^k (i.e. $j \in \{0, 1, ..., k\}$)

Example 4:



Definition 4 (Normal Disc): A properly embedded disc $(D, \partial D) \subseteq (\delta^3, \partial\delta^3)$ is a normal disc if ∂D is transverse to $\Delta^{(1)}$, ∂D meets each edge of $\Delta^{(1)}$ at most once once and $D \cap \Delta^{(0)} = \emptyset$.

Example 5:



Definition 5 (Normal Isotopy): An isotopy $F : M \times I \to M : (m, x) \mapsto F(m, x)$ is normal with respect to the triangulation

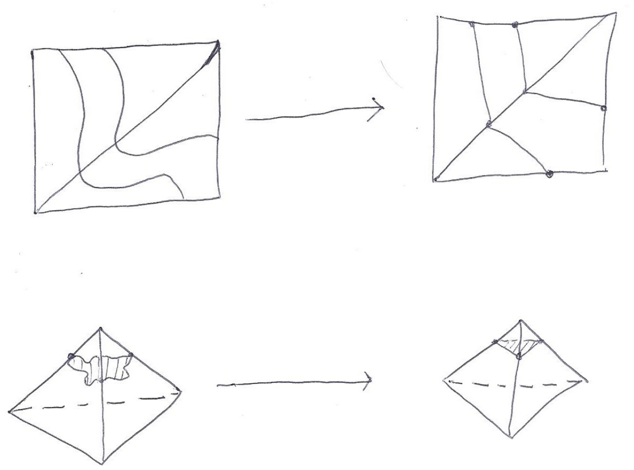
$$\rho : \tilde{\Delta} \to M$$

if for all $x \in I$ the homeomorphism

$$F_x : M \to M : m \mapsto F_x(m) := F(m, x)$$

preserves $M^{(k)}$ for all $k \in \{0, 1, 2, 3\}$ and $F_0 = \text{id}$.

Example 6:



Exercise 3: Show that

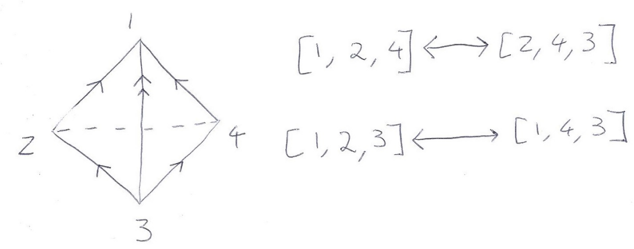
(i) in Δ^2 there are only 3 normal arcs up to normal isotopy,

(ii) in Δ^3 there are only 7 normal discs up to normal isotopy.

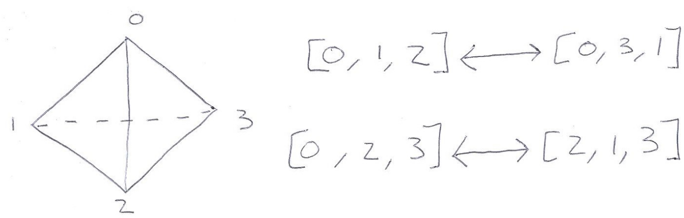
Definition 6 (Normal Embedded Surface): Let F be an embedded surface in M. Then, F is normal if $\rho^{-1}(F)$ is a pairwise disjoint collection of normal discs.

Exercise 4: Determine whether the following are 3-manifolds

(i)



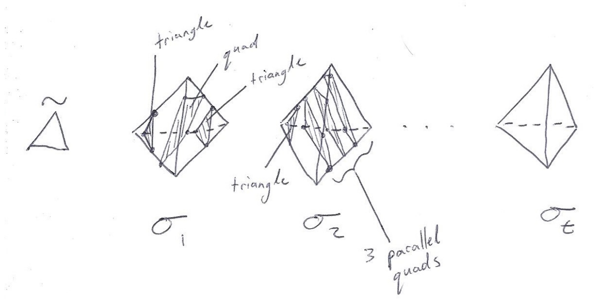
(ii)



Aside: If M is a closed manifold then $v \in M^{(0)}$ implies $\partial(N(v)) \cong S^2$.

More on normal surfaces

Let $F \subseteq M$ be a normal surface. Furthermore, let F be compact so that F is a finite union of normal discs.



Record the number of discs of each disc type

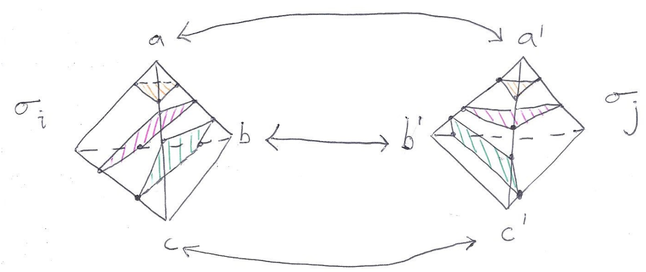
$$v(F) = (t_0^i, t_1^i, t_2^i, t_3^i, q_{01}^i, q_{02}^i, q_{03}^i)$$

$$= (t_0^i, t_1^i, t_2^i, t_3^i, q_{23}^i, q_{13}^i, q_{12}^i) \in \mathbb{Z}_{\geq 0}^{7t}$$

Note that the above uses the equalities $q_{01}^i = q_{23}^i, q_{02}^i = q_{13}^i$ and $q_{03}^i = q_{12}^i$.

Get one linear equation for each triangle in $M^{(2)}$ that is not in ∂M

$$\boxed{t_a^i + q_{bc}^i = t_{a'}^j + q_{b'c'}^j}$$



The above boxed equation is called a matching equation.

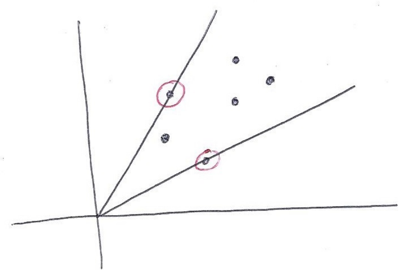
So, $v(F)$ satisfies all matching equations by construction. Furthermore, $v(F)$ satisfies the "quadrilateral constraints"; namely, at most one of the quadrilateral coordinates is non-zero for each 3-simplex.

Theorem 2 [Haken]: For any $v \in \mathbb{Z}^{7t}_{\geq 0}$ that satisfies the matching equations (linear equations) and quadrilateral constraints there is a normal surface F, unique up to normal isotopy, such that $v(F) = v$.

The theorem above is a good result since we can make it algorithmic.

Remark 3: Coordinates of normal surfaces lie in a polyhedral cone in $\mathbb{R}^{7t}_{\geq 0}$ and each such coordinate is:

- a rational linear combination of "extremal solutions"
- an integral linear combination of "fundamental solutions"



Haken's blueprint from the 1960s

Problem: Does M^3 have property P?

Examples:

- P=irreducible
- P=contains a π_1-injective surface

Approach: Reduce to surfaces then use linear programming.

1. Reduce "M has property P" to "some surface in M has property Q".
2. If a surface in M has property Q, then some normal surface in M has property Q.
3. If a normal surface in M has property Q, then a normal surface in a finite constructable set has property Q.
4. Construct an algorithm to decide whether a normal surface has property Q.

Examples:

- Haken 1961: Constructed an algorithm to decide whether a given knot, represented by a knot diagram, is equivalent to the unknot (unknot recognition).
- Jaco-Oertal 1984: Constructed an algorithm to decide if a 3-manifold M contains a π_1-injective surface ("incompressible surface").
- Rubinstein-Thompson 1995: Constructed an algorithm to decide if a 3-manifold is homeomorphic to the 3-sphere. That is, is $M \cong S^3$?