On character varieties:

Surfaces associated to mutation & deformation of hyperbolic 3–manifolds

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Abstract

This thesis investigates the relationship between essential surfaces in a 3–manifold and ideal points of its $SL_2(\mathbb{C})$– and $PSL_2(\mathbb{C})$–character varieties. In 1983 Culler and Shalen introduced a procedure which associates essential surfaces to ideal points of the character variety. This thesis aims to describe associated surfaces in two settings. The first uses the topological notion of mutation, and the second uses deformations of ideal triangulations of cusped hyperbolic 3–manifolds.

A mutation is performed by cutting a manifold along a surface which admits a canonical involution, and regluing via this involution. The first part of this thesis is concerned with the effect of mutation on the character varieties and the conditions under which a mutation surface can be associated to an ideal point of the character variety. This approach led to the discovery of the first known example of so–called *holes* in the variety defined by the A–polynomial of the Kinoshita–Terasaka knot.

To deal with multi–cusped manifolds, the A–polynomial of a 1–cusped manifold is generalised to an *eigenvalue variety.* Boundary curves of essential surfaces arising at ideal points of the character variety are called *strongly detected.* The set of strongly detected boundary curves is determined in terms of Bergman's *logarithmic limit set*, which describes the exponential behaviour of a variety at infinity.

The above result relating to eigenvalue varieties only concerns the boundary curves. Following an approach sketched by Thurston in 1981, normal surfaces are associated to degenerations of ideal triangulations of 3–manifolds. Moreover, an algorithm to compute the boundary curves of a normal surface in an ideal triangulation is given.

This thesis concludes with a description of normal surfaces in the Whitehead link complement, and a computation of the logarithmic limit sets of its deformation and its eigenvalue varieties.

This is to certify that

1. this thesis comprises only my original work towards the PhD except where indicated in the Preface,
2. due acknowledgement has been made in the text to all other material used,
3. this thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Stephan Tillmann

Preface

Chapter 1 largely reviews known results. The only possibly new contributions are the computation of the A–polynomials and the $\overline{A}$–polynomials of torus knots.

The major original contributions of this thesis are contained in Chapters 2 to 5. Chapter 2 builds on Cooper and Long [**13**] and Ruberman [**39**]. Chapter 3 was stimulated by looking at Lash [**32**]. The investigation in Chapter 4 is motivated by a lecture by William Thurston, as well as Kang [**31**] and Weeks [**50**], and has been suggested to me by Craig Hodgson.

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Introduction

In order to study 3–manifolds I use various affine algebraic sets, which are related to representations of the fundamental group into $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$. These target groups have a special place in the current theory, since $PSL_2(\mathbb{C})$ is isomorphic to the group of orientation preserving isometries of hyperbolic 3–space. I am interested in extracting geometric and topological information about a 3–manifold from the study of representations.

In particular, I investigate the relationship between essential surfaces and ideal points of character varieties. Character varieties allow the study of representations into $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ up to conjugacy. Culler and Shalen introduced a procedure in [**18**] which associates an action of a finitely generated group Γ on a tree with an ideal point of a curve in the $SL_2(\mathbb{C})$–character variety. This construction provides a link between the geometry of a manifold and its topology, since $SL_2(\mathbb{C})$–representations of fundamental groups of connected manifolds are related to hyperbolic structures, and actions of the fundamental group on trees are related to essential surfaces. This link, however, is rather mysterious since it uses valuation theory, Bass–Serre theory and a construction by Stallings which involves the choice of a triangulation of the manifold and the choice of a map from its universal cover to a tree.

If one hopes that every representation into $SL_2(\mathbb{C})$ carries some geometric meaning, one may also be led to hope that an ideal point corresponds to a geometric splitting of a manifold associated to the limiting representation. This is the spirit in which my research is conducted and examples are analysed. I am interested in associating surfaces to ideal points of the character variety in two settings. In the first setting I use the topological notion of mutation, and in the second setting I use deformations of ideal triangulations of 3–manifolds. There is a surface in the manifold $m137$ (in `SnapPea` notation), which is detected by both approaches, and there is a surface in the complement of the figure eight knot which is not detected by either of these approaches.

The contents of this thesis is as follows:

Chapter 1: Culler–Shalen theory and other related background is discussed, and the terminology and notation used throughout this thesis is introduced. Moreover, the A–polynomials and the $\overline{A}$–polynomials of torus knots are computed, and a surface in the manifold $m137$ is shown to be associated to an ideal point in its character variety.

Chapter 2: This chapter builds on Cooper and Long [**13**] and Ruberman [**39**]. Let M be an irreducible 3–manifold and S be a properly embedded surface in M. Assume that S admits a canonical involution τ. Then M^τ is the manifold obtained by cutting M along S and regluing via τ. This process is called a *mutation.* Mutative manifolds are generally difficult to distinguish. A birational map between subvarieties in the character varieties of M and M^τ is described. By studying the birational map, one can decide in certain circumstances whether a mutation surface is detected by an ideal point. As an application of this approach, the first known example of so–called "holes" in the variety defined by the A–polynomial is given in the section on the Kinoshita–Terasaka knot.

Chapter 3: The set of ideal points of algebraic varieties over $\mathbb{C} - \{0\}$ is defined via Bergman's *logarithmic limit set*, which describes the exponential behaviour of a variety at infinity. Subsequently, the A–polynomial of a manifold whose boundary consists of a single torus is generalised to an eigenvalue variety of a manifold whose boundary consists of a finite number of tori. Boundary curves of essential surfaces arising at ideal points of the character variety are called *strongly detected.* The set

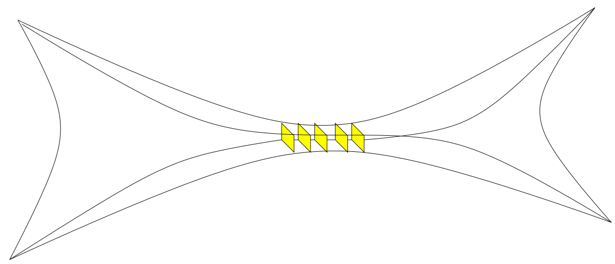


FIGURE 0.1. A degenerating tetrahedron

of strongly detected boundary curves is determined in terms of the logarithmic limit set. This enables one to read off the detected boundary curves of a multi–cusped manifold in a similar way to the 1–cusped case, where the slopes are encoded in the Newton polygon of the A–polynomial. This work was stimulated by looking at Lash [**32**].

Chapter 4: The investigation in this chapter is motivated by the following outline given in a lecture by William Thurston in 1981 for an orientable cusped complete hyperbolic 3–manifold of finite volume with an ideal hyperbolic triangulation, and was suggested to me by Craig Hodgson. There is an algebraic variety associated to an ideal triangulation, which describes the shapes of ideal hyperbolic tetrahedra. As the hyperbolic structure degenerates, an ideal point of this variety is approached. Thurston observed that the tetrahedra associated to the degeneration become very long and thin, and the manifold will split apart. In certain cases, a surface is expected to develop in the thin part, along which this splitting occurs. This behaviour is described when the ratios of logarithms of moduli of the shape parameters approach rational values. In this case, the splitting surface arises as a normal surface with respect to the ideal triangulation. We investigate the relationship between these surfaces and the normal sufaces arising from normal surface Q–theory, and give an algorithm to compute the boundary curves of a normal surface in an ideal triangulation, where we rely on work and ideas by Kang [**31**] and Weeks [**50**].

Chapter 5: Normal surfaces in the Whitehead link complement are described, and the logarithmic limit sets of its deformation and its eigenvalue varieties are computed.

Acknowledgements: First and foremost, I thank my supervisors Craig Hodgson and Walter Neumann for their constant support and inspiration during the last four years, and for showing me the beauty and fascination of the field of mathematics that I am now working in. (I also thank them for all the travel, food and wine.)

I thank many members of the Department of Mathematics and Statistics in Melbourne, both administrative and academic staff, for making Melbourne such a warm and friendly place, and a great working environment. The list of names would grow too long if I tried to mention everyone, and I would be too afraid to forget someone. So thank you all!

Last, I thank my beautiful friends (who are all over the place) and my family (who are also scattered) for their lasting friendship, support and inspiration.

CHAPTER 1

Preliminaries

This chapter discusses some aspects of Culler–Shalen theory and related background which will be used in the subsequent chapters, without giving a historical or otherwise complete account of its place in the field. Excellent references to Culler–Shalen theory are Shalen's articles [**42, 43**], as well as the paper [**6**] by Boyer and Zhang.

1.1. Algebraic varieties associated to 3–manifolds

This section consists mostly of definitions of various affine algebraic sets and some of their basic properties. It also contains results concerning the A–polynomials and the $\overline{A}$–polynomials of torus knots.

1.1.1. Representation variety. Let M be a 3–manifold. A representation of $\pi_1(M)$ into $SL_2(\mathbb{C})$ is a homomorphism $\rho : \pi_1(M) \to SL_2(\mathbb{C})$, and the set of representations is $\mathfrak{R}(M) = \text{Hom}(\pi_1(M), SL_2(\mathbb{C}))$. This set is often called the *representation variety* of M.

If $\langle \gamma_1, \dots, \gamma_n \mid r_j \rangle$ is a presentation for the fundamental group $\Gamma = \pi_1(M)$, then a representation is uniquely determined by the point $(\rho(\gamma_1), \dots, \rho(\gamma_n)) \in SL_2(\mathbb{C})^n \subset \mathbb{C}^{4n}$. The latter inclusion introduces affine coordinates. Substituting n general matrices into the relators gives sets of polynomial relations in these affine coordinates, and the Hilbert basis theorem implies that $\mathfrak{R}(M)$ inherits the structure of an affine algebraic set.

Two representations are *equivalent* if they differ by an inner automorphism of $SL_2(\mathbb{C})$. For each $\rho \in \mathfrak{R}(M)$, its character is the function $\chi_\rho : \Gamma \to \mathbb{C}$ defined by

$\chi_\rho(\gamma) = \operatorname{tr}\rho(\gamma)$. It follows that equivalent representations have the same character since the trace is invariant under conjugation.

A representation is *irreducible* if the only subspaces of $\mathbb{C}^2$ invariant under its image are trivial. This is equivalent to saying that the representation cannot be conjugated to a representation by upper triangular matrices. Otherwise a representation is *reducible*. The following facts will often be used implicitly:

LEMMA 1.1. [**18**]

1. *Let $\rho \in \mathfrak{R}(M)$. Then ρ is reducible if and only if $\chi_\rho(c) = 2$ for each element c of the commutator subgroup $\Gamma' = [\Gamma, \Gamma]$ of Γ.*
2. *Let $\rho, \sigma \in \mathfrak{R}(M)$ satisfy $\chi_\rho = \chi_\sigma$ and assume that ρ is irreducible. Then ρ and σ are equivalent.*
3. *Let V be an irreducible component of $\mathfrak{R}(M)$. Then any representation equivalent to a representation in V must itself belong to V.*

The above lemma implies that irreducible representations are determined by characters up to equivalence, and the reducible representations form a closed subset of $\mathfrak{R}(M)$.

1.1.2. Character variety. The collection of characters $\mathfrak{X}(M)$ turns out to be an affine algebraic set, which is called the *character variety.* There is a regular map $\mathrm{t} : \mathfrak{R}(M) \to \mathfrak{X}(M)$ taking representations to characters. According to [**23**], affine coordinates of $\mathfrak{X}(M)$ can be chosen as follows. Number the words

$$\{\gamma_i\gamma_j \mid 1 \le i < j \le n\} \cup \{\gamma_i\gamma_j\gamma_k \mid 1 \le i < j < k \le n\}$$

starting from $n+1$ onwards and denote them accordingly by $\gamma_{n+1}, \dots, \gamma_m$. Then a character is uniquely determined by the point $(\operatorname{tr}\rho(\gamma_1), \dots, \operatorname{tr}\rho(\gamma_m)) \in \mathbb{C}^m$, where $m = n + \binom{n}{2} + \binom{n}{3} = \frac{n(n^2+5)}{6}$.

1.1.3. Tautological representation. Let V be an irreducible subvariety of $\mathfrak{X}(M)$. By [**18**], there is an irreducible subvariety $R_V \subset \mathfrak{R}(M)$ such that $\mathrm{t}(R_V) =$

V. The function field $F = \mathbb{C}(R_V)$ contains $K = \mathbb{C}(V)$. We now obtain the *tautological representation* $\mathcal{P} : \Gamma \to SL_2(F)$ defined by

$$\mathcal{P}(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where the identity } \rho(\gamma) = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix} \text{ for all } \rho \in R_V$$

determines the functions $a, b, c, d \in F$. One can think of this construction as restricting the coordinate functions to R_V.

For each $\gamma \in \Gamma$ define $I_\gamma = \operatorname{tr} \mathcal{P}(\gamma) \in K \subset F$. It follows from the definition of the tautological representation that $I_\gamma(\rho) = \operatorname{tr} \rho(\gamma) \in \mathbb{C}$ for all $\rho \in R_V$, and hence we have a function $I_\gamma : R_V \to \mathbb{C}$. Since $I_\gamma \in K$, it may also be thought of as a function on V.

More generally, for each $\gamma \in \pi_1(M)$, we define a function $I_\gamma : \mathfrak{R}(M) \to \mathbb{C}$ by $I_\gamma(\rho) = \operatorname{tr} \rho(\gamma)$. Then I_γ is an element in the coordinate ring $\mathbb{C}[\mathfrak{R}(M)]$ (see [**42**]).

1.1.4. Projective representations. There is also a notion of character variety arising from representations into $PSL_2(\mathbb{C})$, and the relevant objects are denoted by placing a bar over the previous notation. The natural map $\mathrm{q} : \mathfrak{X}(M) \to \overline{\mathfrak{X}}(M)$ is finite–to–one, but in general not onto. It is the quotient map corresponding to the $H^1(\Gamma; \mathbb{Z}_2)$–action on $\mathfrak{X}(M)$, where $H^1(\Gamma; \mathbb{Z}_2) = \operatorname{Hom}(\Gamma, \mathbb{Z}_2)$. This action is not free in general.

Central extensions of $PSL_2(\mathbb{C})$ and Γ by $\mathbb{Z}_2$ must be studied in order to decide whether a representation into $PSL_2(\mathbb{C})$ lifts to a representation into $SL_2(\mathbb{C})$. In [**6**], Boyer and Zhang give examples of (non–hyperbolic) 3–manifolds where $\dim_{\mathbb{C}} \mathfrak{X}(M) = 0$, but $\dim_{\mathbb{C}} \overline{\mathfrak{X}}(M) = 1$.

As with the $SL_2(\mathbb{C})$–character variety, there is a surjective "quotient" map $\bar{\mathrm{t}} : \overline{\mathfrak{R}}(M) \to \overline{\mathfrak{X}}(M)$, which is constant on conjugacy classes, and with the property that if $\overline{\rho}$ is an irreducible representation, then $\bar{\mathrm{t}}^{-1}(\bar{\mathrm{t}}(\overline{\rho}))$ is the orbit of $\overline{\rho}$ under conjugation. Again, a representation is irreducible if it is not conjugate to a representation by upper triangular matrices.

A statement similar to the description of $SL_2(\mathbb{C})$–characters is true. Let $\mathfrak{F}_n$ be the free group on $\xi_1, \ldots, \xi_n$, and let $y_1, \ldots, y_m$ be the $\frac{n(n^2+5)}{6}$ elements of $\mathfrak{F}_n$ corresponding to the single generators and ordered double and triple products thereof.

LEMMA 1.2. [**6**] *Suppose that* Γ *is generated by* $\gamma_1, \ldots, \gamma_n$ *and that* $\overline{\rho}, \overline{\rho}' \in \overline{\mathfrak{R}}(\Gamma)$. *Choose matrices* $A_1, \ldots, A_n, B_1, \ldots, B_n \in SL_2(\mathbb{C})$ *satisfying* $\overline{\rho}(\gamma_i) = \pm A_i$ *and* $\overline{\rho}'(\gamma_i) = \pm B_i$ *for each* i. *Define* $\rho, \rho' \in \mathfrak{R}(\mathfrak{F}_n)$ *by requiring that* $\rho(\xi_i) = A_i$ *and* $\rho'(\xi_i) = B_i$ *for each* $i \in \{1, \ldots, n\}$. *Then* $\chi_{\overline{\rho}} = \chi_{\overline{\rho}'}$ *if and only if there is a homomorphism* $\epsilon \in \mathrm{Hom}(\mathfrak{F}_n, \{\pm 1\})$ *for which* $\mathrm{tr}\,\rho'(y_j) = \epsilon(y_j)\,\mathrm{tr}\,\rho(y_j)$ *for each* $j \in \{1, \ldots, m\}$.

Boyer and Zhang introduce the tautological representation associated to an irreducible component of $\overline{\mathfrak{R}}(\Gamma)$ in [**6**], which is analogous to the $SL_2(\mathbb{C})$–version, and will be denoted by $\overline{\mathcal{P}}$.

1.1.5. Dehn surgery component. If M admits a complete hyperbolic structure of finite volume, then there is a discrete and faithful representation $\pi_1(M) \to PSL_2(\mathbb{C})$. This representation is neccessarily irreducible, as hyperbolic geometry otherwise implies that M has infinite volume.

If M is not compact, then a *compact core* of M is a compact manifold $\overline{M}$ such that M is homeomorphic to the interior of $\overline{M}$. We will rely heavily on the following result:

THEOREM 1.3 (Thurston). [**44, 42**] *Let* M *be a complete hyperbolic manifold of finite volume with* h *cusps, and let* $\overline{\rho}_0 : \pi_1(M) \to PSL_2(\mathbb{C})$ *be a discrete and faithful representation associated to the complete hyperbolic structure. Then* $\overline{\rho}_0$ *admits a lift* ρ_0 *into* $SL_2(\mathbb{C})$ *which is still discrete and faithful. The (unique) irreducible component* X_0 *in the* $SL_2(\mathbb{C})$*–character variety containing the character* χ_0 *of* ρ_0 *has (complex) dimension* h.

Furthermore, if $T_1, \ldots, T_h$ are the boundary tori of a compact core of M, and if γ_i is a non–trivial element in $\pi_1(M)$ which is carried by T_i, then $\chi_0(\gamma_i) = \pm 2$ and χ_0 is an isolated point of the set

$$X^* = \{\chi \in X_0 \mid I^2_{\gamma_1} = \ldots = I^2_{\gamma_h} = 4\}.$$

The respective irreducible components containing the so–called *complete representations* ρ_0 and $\overline{\rho}_0$ are denoted by $\mathfrak{R}_0(M)$ and $\overline{\mathfrak{R}}_0(M)$ respectively. In particular, $\mathrm{t}(\mathfrak{R}_0) = \mathfrak{X}_0$ and $\bar{\mathrm{t}}(\overline{\mathfrak{R}}_0) = \overline{\mathfrak{X}}_0$ are called the respective *Dehn surgery components* of the character varieties of M, since the holonomy representations of hyperbolic manifolds or orbifolds obtained by performing high order Dehn surgeries on M are near $\overline{\rho}_0$ (see [**44**]).

1.1.6. A–polynomial. The A–polynomial was introduced in [**11**], and we define it following [**13**]. Let M be a manifold with boundary consisting of a single torus, and choose a generating set $\{\mathcal{M}, \mathcal{L}\}$ for the fundamental group of ∂M. The elements $\mathcal{M}$ and $\mathcal{L}$ will be referred to as meridian and longitude respectively.

Let $\mathfrak{R}_U(M)$ be the subvariety of $\mathfrak{R}(M)$ defined by two equations which specify that the lower left entries in $\rho(\mathcal{M})$ and $\rho(\mathcal{L})$ are equal to zero. Any representation in $\mathfrak{R}(M)$ is conjugate to a representation in $\mathfrak{R}_U(M)$ since two commuting matrices in $SL_2(\mathbb{C})$ have a common invariant subspace. We define an *eigenvalue map* from $\mathfrak{R}_U(M)$ to $(\mathbb{C} - \{0\})^2$ by taking an element ρ of $\mathfrak{R}_U(M)$ to the upper left entries of $\rho(\mathcal{M})$ and $\rho(\mathcal{L})$. Taking the closure of the image of this map and discarding zero–dimensional components, one obtains the *eigenvalue variety*, which is necessarily defined by a principal ideal. A generator for the radical of this ideal is called the *A–polynomial*. After fixing a basis for the boundary torus, the A-polynomial is well defined up to multiplication by units in $\mathbb{C}[l^{\pm 1}, m^{\pm 1}]$, and it follows from [**11**] and [**15**] that the constant multiple can be chosen such that the coefficients are all integers with greatest common divisor equal to one.

In general, one needs a computer to calculate the A–polynomial. It is shown in [**11**] that if $\mathfrak{k}$ is a nontrivial (p, q)–torus knot, then $A_{\mathfrak{k}}(l, m)$ is divisible by $lm^{pq} + 1$

(where we follow the convention that if M is the complement of a knot $\mathfrak{k}$ in S^3, then $\{\mathcal{M}, \mathcal{L}\}$ is a standard peripheral system).

PROPOSITION 1.4. [1] *Let $\mathfrak{k}$ be a (p,q)–torus knot. If $p = 2$ or $q = 2$, then $A_{\mathfrak{k}}(l,m) = (l-1)(lm^{pq}+1)$, otherwise $A_{\mathfrak{k}}(l,m) = (l-1)(lm^{pq}+1)(lm^{pq}-1)$.*

PROOF. The fundamental group of $S^3 - \mathfrak{k}$ is presented by $\Gamma = \langle u, v \mid u^p = v^q \rangle$, and a standard peripheral system is given by $\mathcal{M} = u^n v^m$ and $\mathcal{L} = u^p \mathcal{M}^{-pq}$, where $mp + nq = 1$. These facts can be found in [**9**] on page 45. The factor $l - 1$ arises from reducible representations, and all factors arising from components containing irreducible representations are to be determined.

The element u^p is in the centre of Γ, since it is identical to v^q and hence commutes with both generators. Thus the image of u^p is in the center of $\rho(\Gamma)$. Since two commuting elements have a common eigenvector, $\rho(u)^p$ has a common eigenvector with $\rho(u)$ and $\rho(v)$ respectively. If the representation is irreducible, these eigenvectors have to be distinct. Thus, after conjugation it may be assumed that the generators map to upper and lower triangular matrices respectively, and the central element is represented by a diagonal matrix. Direct matrix computations show that the commutativity with either of the generators gives $\rho(u)^p = \pm E$.

Assume that $\rho(u)^p = -E$. The relation $\rho(\mathcal{L}) = -\rho(\mathcal{M})^{-pq}$, which is equivalent to $\rho(\mathcal{L}\mathcal{M}^{pq}) = -E$, then implies the equation $lm^{pq} = -1$. This is the curve obtained in [**11**] by sending u and v to noncommuting elements of $SL_2(\mathbb{C})$ of order exactly $2p$ and $2q$ respectively.

If the image of $\rho(u)^p$ is trivial, then $\rho(v)^q = \rho(u)^p = E$. If p or q equals 2, this implies that the image of one of the generators is $\pm E$. But this yields that $\rho(\Gamma)$ is abelian, and therefore contradicts the irreducibility assumption. This completes the proof of the first assertion.

[1] I thank Walter Neumann for pointing out that a proof contained in [**47**] could be used to compute the A–polynomials of torus knots.

Now assume that neither of p and q is equal to two. As above, curves of irreducible representations are obtained by sending u and v to noncommuting elements of $SL_2(\mathbb{C})$ of order exactly p and q respectively. Any of these curves yields a component of the eigenvalue variety defined by the equation $lm^{pq} = 1$, and this finishes the proof of the proposition. In fact, with a more precise analysis one could count the number of 1–dimensional curves in the character variety of a torus knot. ∎

1.1.7. $\overline{\mathbf{A}}$–polynomial. Analogous to the $SL_2(\mathbb{C})$–eigenvalue variety, there is an eigenvalue variety associated to the $PSL_2(\mathbb{C})$–character variety. This is the variety containing points corresponding to pairs of squares of eigenvalues of the meridian and longitude (in the case where the boundary of M consists of a single torus). A generator for its defining (principal) ideal is the $\overline{A}$–polynomial, and its variables have the unfortunate names L and M. Thus, $\overline{A}_M(L, M)$ is the $\overline{A}$–polynomial associated to the manifold M.

PROPOSITION 1.5. *Let $\mathfrak{k}$ be a (p, q)–torus knot.*
Then $\overline{A}_{\mathfrak{k}}(L, M) = (L-1)(LM^{pq}-1)$.

PROOF. It follows from the lemma in Section 6 of [**11**] that all $PSL_2(\mathbb{C})$–representations of a knot group lift to $SL_2(\mathbb{C})$. Thus, all factors of the A–polynomial arise from factors of the $\overline{A}$–polynomial and vice versa.

The eigenvalue of the meridian appears with even powers in the (unfactorised) A–polynomial of $\mathfrak{k}$, and the sign of the longitude's eigenvalue is uniquely determined since $\mathcal{L}$ is null–homologous. Thus, $A_{\mathfrak{k}}(l, m) = A_{\mathfrak{k}}(l, -m)$, but $A_{\mathfrak{k}}(l, m) = 0$ does not imply $A_{\mathfrak{k}}(-l, m) = 0$ in general. However, since $\overline{A}_{\mathfrak{k}}(l^2, m^2) = \overline{A}_{\mathfrak{k}}((-l)^2, m^2)$, it follows that the variety defined by $\overline{A}_{\mathfrak{k}}(l^2, m^2) = 0$ is the locus of $A_{\mathfrak{k}}(l, m)A_{\mathfrak{k}}(-l, m) = 0$. Expanding $A_{\mathfrak{k}}(l, m)A_{\mathfrak{k}}(-l, m) = 0$, substituting $l^2 = L$ and $m^2 = M$ and deleting repeated factors therefore gives the result as stated. ∎

1.1.8. Computing character varieties. In this subsection a cross–section for the quotient map from the representation space to the character variety is defined in the case of 2–generator groups.

Let Γ be an arbitrary finitely generated group. It follows from Lemma 1.1, that the set $\mathfrak{Red}(\Gamma)$ consisting of reducible representations is a subvariety of $\mathfrak{R}(\Gamma)$. Let $\mathfrak{R}^i(\Gamma)$ denote the closure of the set of irreducible representations. According to [**18**], the images $\mathfrak{X}^r(\Gamma) = \mathrm{t}(\mathfrak{Red}(\Gamma))$ and $\mathfrak{X}^i(\Gamma) = \mathrm{t}(\mathfrak{R}^i(\Gamma))$ are closed algebraic sets. Then $\mathfrak{X}^r(\Gamma) \cup \mathfrak{X}^i(\Gamma) = \mathfrak{X}(\Gamma)$, and the union may or may not be disjoint. Note that $\mathfrak{X}^r(\Gamma)$ is completely determined by the abelianisation of Γ, since the character of any reducible non–abelian representation is also the character of an abelian representation. It is shown in [**18**] that fibres of $\mathrm{t} : \mathfrak{R}^i(\Gamma) \to \mathfrak{X}(\Gamma)$ have dimension three.

Suppose that Γ is a 2–generator group with presentation $\langle \gamma, \delta \mid r_i \rangle$. Let ρ be an irreducible representation in $\mathfrak{R}(\Gamma)$. There are four choices of bases $\{b_1, b_2\}$ for $\mathbb{C}^2$ with respect to which ρ has the form:

$$\rho(\gamma) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\delta) = \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix}. \tag{1.1}$$

These bases can be obtained by choosing b_1' invariant under $\rho(\gamma)$, b_2' invariant under $\rho(\delta)$, and then adjusting by a matrix which is diagonal with respect to $\{b_1', b_2'\}$. Thus, any irreducible representation in $\mathfrak{R}^i(\Gamma)$ is conjugate to a representation in the subvariety $\mathfrak{C}(\Gamma) \subseteq \mathfrak{R}^i(\Gamma)$ defined by two equations which specify that the lower left entry in the image of γ and the upper right entry in the image of δ are equal to zero, and an additional equation which specifies that the upper right entry in the image of γ equals one. It follows from the construction that the restriction $\mathrm{t} : \mathfrak{C}(\Gamma) \to \mathfrak{X}(\Gamma)$ is generically 4–to–1, and corresponds to the action of the Kleinian four group on the set of possible bases for the normal form (1.1). The involutions $(s, t, u) \to (s^{-1}, t^{-1}, u)$ and $(s, t, u) \to (s, t^{-1}, u + (s - s^{-1})(t - t^{-1}))$ generate this group.

$\mathfrak{C}(\Gamma)$ may be thought of as a variety in $(\mathbb{C}-\{0\})^2\times\mathbb{C}$, and the intersection of $\mathfrak{C}(\Gamma)$ with $\mathfrak{Red}(\Gamma)$ corresponds to the intersection with the hyperplane $\{u=0\}$. For any reducible non–abelian representation $\sigma\in\mathfrak{R}(\Gamma)$, there is a representation $\rho\in\mathfrak{C}(\Gamma)$ with $u=0$ such that $\chi_\sigma=\chi_\rho$. Moreover, σ is conjugate to a representation in $\mathfrak{C}(\Gamma)$ unless $\sigma(\delta)$ is a non–trivial parabolic.

Consider the "conjugation" map $c:\mathfrak{C}(\Gamma)\times SL_2(\mathbb{C})\to\mathfrak{R}(\Gamma)$ defined by $c(\rho,X)=X^{-1}\rho X$. This is a regular map, and we have $\overline{c(\mathfrak{C}(\Gamma)\times SL_2(\mathbb{C}))}=\mathfrak{R}^i(\Gamma)$. Furthermore, if $V\subset\mathfrak{C}(\Gamma)$ is an irreducible component, then $\overline{c(V)}\subset\mathfrak{R}(\Gamma)$ is irreducible. It is convenient to work with $\mathfrak{C}(\Gamma)\subset\mathfrak{R}^i(\Gamma)$ in some applications of Culler–Shalen theory, and we therefore summarise its properties:

LEMMA 1.6. *Let* $\Gamma=\langle\gamma,\delta\mid r_i(\gamma,\delta)=1\rangle$ *be a 2–generator group. The variety* $\mathfrak{C}(\Gamma)$ *defined in* $(\mathbb{C}-\{0\})^2\times\mathbb{C}$ *by (1.1) and the polynomial equations arising from* $r_i(\rho(\gamma),\rho(\delta))=E$ *defines a 4–to–1 (possibly branched) cover of* $\mathfrak{X}^i(\Gamma)$.

We remark that $\mathfrak{C}(\Gamma)$ is defined up to polynomial isomorphism once an unordered generating set has been chosen.

1.2. Surfaces and ideal points

In this section, it is described how Culler and Shalen associate essential surfaces to ideal points of curves in the character variety. There is a related construction by Morgan and Shalen in a sequence of papers starting with [**35**], which generalises the work of [**18**] to higher dimensional components of $\mathfrak{X}(M)$. This more general setting is not needed for the purpose of this thesis, since the ideal points under consideration can always be assumed to be ideal points of a curve in $\mathfrak{X}(M)$ or $\overline{\mathfrak{X}}(M)$ respectively.

1.2.1. The following material can be found in [**41**]. Two varieties V,W are *birationally equivalent* if there are rational maps $\varphi:V\to W$ and $\psi:W\to V$ such that $\varphi(V)$ is dense in W, $\psi(W)$ is dense in V, and $\varphi\circ\psi=1$ as well as $\psi\circ\varphi=1$ where defined.

The map $J : \mathbb{C}^m \to \mathbb{C}P^m$ defined by $J(z_1, \ldots, z_m) = [1, z_1, \ldots, z_m]$ is a diffeomorphism, and if $V \subset \mathbb{C}^m$ is a variety, then $\overline{J(V)}$ is termed a *projective completion* of V. In case that V is a 1–dimensional irreducible variety, there is a unique non–singular projective variety $\tilde{V}$ which is birationally equivalent to $\overline{J(V)}$. $\tilde{V}$ is called the *smooth projective completion* of V, and the *ideal points* of $\tilde{V}$ are the points of $\tilde{V}$ corresponding to $\overline{J(V)} - J(V)$ under the birational equivalence. Moreover, the function fields of V and $\tilde{V}$ are isomorphic.

1.2.2. Ideal points. Let $C \subset \mathfrak{X}(M)$ be a curve, i.e. a 1–dimensional irreducible subvariety, and denote its smooth projective completion by $\tilde{C}$. We will refer to the ideal points of $\tilde{C}$ also as ideal points of C. The function fields of C and $\tilde{C}$ are isomorphic. Denote them by K. Any ideal point ξ of $\tilde{C}$ determines a (normalised, discrete, rank 1) valuation ord_ξ of K, by

$$ord_\xi(f) = \begin{cases} k & \text{if } f \text{ has a zero of order } k \text{ at } \xi \\ \infty & \text{if } f = 0 \\ -k & \text{if } f \text{ has a pole of order } k \text{ at } \xi \end{cases}$$

Note that $ord_\xi(z) = 0$ for all non-zero constant functions $z \in \mathbb{C}$. In the language of algebraic geometry, the valuation ring $\{f \in K \mid ord_\xi(f) \geq 0\}$ of ord_ξ is the local ring at ξ.

Let $R_C \subset \mathfrak{R}(M)$ be an irreducible subvariety such that $\mathrm{t}(R_C) = C$. The function field $F = \mathbb{C}(R_C)$ contains $K = \mathbb{C}(C)$, and the extension theorem for valuations (see [**42**]) implies that there is an integer $d > 0$ and a valuation $v : F^* \to \mathbb{Z}$ such that $v|_{K^*} = d \cdot ord_\xi$. Also recall that we have a tautological representation $\mathcal{P} : \pi_1(M) \to SL_2(F)$.

1.2.3. Serre's tree. We continue in the notation of the previous subsection. Associated to the field F and valuation v on F, Serre has defined a tree $\mathcal{T}_v$ on which $SL_2(F)$ acts simplicially without inversions. For details see [**40, 2, 42**].

Let $\mathcal{O} = \{a \in F \mid v(a) \geq 0\}$ be the *valuation ring*, and π be an *uniformizer*, i.e. an element such that $v(\pi) = 1$. An $\mathcal{O}$–lattice L of F^2 is any $\mathcal{O}$–submodule of the form $L = \mathcal{O}x + \mathcal{O}y$, where $x, y \in F^2$ are linearly independent over F. The group F^* acts by left multiplication on the set of $\mathcal{O}$–lattices, and orbits of this action give equivalence classes. Denote the equivalence class of L by $\Lambda = [L]$. There is a well-defined distance function d on the set of equivalence classes. Given $\mathcal{O}$–lattices L_1, L_2 there is $m \in \mathbb{Z}$ such that $\pi^m L_2 \subset L_1$. The "basis theorem" for submodules of finitely generated free modules over principal ideal domains asserts that if $\pi^m L_2 \subset L_1$, then there is a basis $\{x, y\}$ for L_1 such that $\{\pi^f x, \pi^g y\}$ is a basis for $\pi^m L_2$ for some $f, g \in \mathbb{Z}$. It turns out that $|f - g|$ is well defined for the equivalence classes $\Lambda_1 = [L_1]$, $\Lambda_2 = [L_2]$. We therefore obtain a well–defined distance function $d(\Lambda_1, \Lambda_2) = |f - g|$. The tree $\mathcal{T}_v$ is obtained from the following sets of vertices and edges:

$$\mathcal{V} = \{\Lambda \mid \Lambda = [L], L \text{ is an } \mathcal{O}\text{–lattice in } V\},$$

$$\mathcal{E} = \{(\Lambda_1, \Lambda_2) \mid d(\Lambda_1, \Lambda_2) = 1\}.$$

We regard the edges as oriented and write $(\Lambda_2, \Lambda_1) = \overline{(\Lambda_1, \Lambda_2)}$.

The group $SL_2(F)$ acts on lattices by $AL = \mathcal{O}Ax + \mathcal{O}Ay$ for $A \in SL_2(F)$, where $L = \mathcal{O}x + \mathcal{O}y$, and this action on lattices is well–defined for equivalence classes of lattices, hence giving an action of $SL_2(F)$ on $\mathcal{T}_v$. We have:

LEMMA 1.7. [**40, 42**]

1. *$SL_2(F)$ acts on $\mathcal{T}_v$ simplicially, without inversions.*
2. *For any vertex Λ of $\mathcal{T}_v$, $Stab(\Lambda)$ is conjugate to $SL_2(\mathcal{O})$.*

The tautological representation $\mathcal{P} : \pi_1(M) \to SL_2(F)$ can be used to pull the action of $SL_2(F)$ back to an action of $\pi_1(M)$ on $\mathcal{T}_v$, and we write $\mathcal{P}(\gamma) \cdot \Lambda = \gamma\Lambda$.

1.2.4. Properties of the action. The notation in this subsection is continued from Subsection 1.2.2. For each $\gamma \in \pi_1(M)$ we have the function I_γ on C, defined by $I_\gamma = \operatorname{tr} \mathcal{P}(\gamma) \in K \subset F$. At an ideal point ξ, it has a well–defined value $I_\gamma(\xi) \in \mathbb{C} \cup \{\infty\}$.

LEMMA 1.8. [**18**] *For any* $\gamma \in \pi_1(M)$ *the following are equivalent:*

1. $I_\gamma(\xi) \in \mathbb{C}$, *i.e.* I_γ *does not have a pole at* ξ.
2. *Some vertex of* $\mathcal{T}_v$ *is fixed by* γ, *where* $v|_{K^*} = d \cdot ord_\xi$.

The action of $\pi_1(M)$ on a tree is called *non–trivial* if $\pi_1(M)$ is not contained in the stabiliser of a vertex. Using the fact that the action of $\pi_1(M)$ arises from an ideal point, the previous lemma implies the following proposition, which shows that there is a non–trivial action without inversions associated to any ideal point of a curve in the character variety:

PROPOSITION 1.9. [**18**] *The action of* $\pi_1(M)$ *on* $\mathcal{T}_v$ *is non–trivial.*

1.2.5. The following can be found in [**40**]. The *translation length* of an element is defined by

$$\ell(\gamma) = \min\{d(\Lambda, \gamma\Lambda) \mid \Lambda \in \mathcal{T}_v\}.$$

Group elements which fix a vertex have translation length equal to zero, and are called *elliptic*. The set of fixed points of an elliptic element is a subtree of $\mathcal{T}_v$. An element with positive translation length is called *loxodromic*. Loxodromic elements act by translations along a unique line, which is called their *axis*, and denoted by $A(\gamma)$. The *invariant set* of an element γ is understood to be its fixed set if it is elliptic and its axis if it is loxodromic.

LEMMA 1.10. [**40**] *Let* Γ *be a finitely generated group which acts simplicially on a tree. Let the generators of* Γ *be* $\gamma_1, \ldots, \gamma_m$, *and assume that* γ_i *and* $\gamma_i\gamma_j$ *have fixed points for all* i, j. *Then* Γ *fixes a vertex.*

1.2.6. Surface associated to the action. A surface *associated to the action* of $\pi_1(M)$ on $\mathcal{T}_v$ is defined by Culler and Shalen using a construction due to Stallings (see [**42**]). If the given manifold is not compact, replace it by a compact core. Choose a triangulation of M and give the universal cover $\tilde{M}$ the induced triangulation, so that the fundamental group of M acts simplicially on this induced triangulation. One can then construct a simplicial, $\pi_1(M)$–equivariant map f from $\tilde{M}$ to $\mathcal{T}_v$. The inverse image of midpoints of edges is a surface in $\tilde{M}$ which descends to a surface S in M. It turns out that since the action of $\pi_1(M)$ on $\mathcal{T}_v$ is non–trivial, if necessary, one can change the map f by homotopy such that the surface S is *essential* in the sense defined below. The associated surface S depends upon the choice of triangulation of M and the choice of the map f.

An associated surface often contains finitely many parallel copies of one of its components. These parallel copies are somewhat redundant, and we implicitly discard them, whilst we still call the resulting surface associated.

1.2.7. Essential surfaces. A *surface* S in a compact 3–manifold M will always mean a 2–dimensional PL submanifold *properly embedded* in M, that is, a closed subset of M with $\partial S = S \cap \partial M$. If M is not compact, we replace it by a compact core. An embedded sphere S^2 in a 3–manifold M is called *incompressible* if it does not bound an embedded ball in M, and a manifold is *irreducible* if it contains no incompressible 2–spheres.

An orientable surface S without 2–sphere or disc components in an orientable 3–manifold M is called *irreducible* if for each disc $D \subset M$ with $D \cap S = \partial D$ there is a disc $D' \subset S$ with $\partial D' = \partial D$. We will also use the following definition:

DEFINITION 1.11. [**42**] *A surface S in a compact, irreducible, orientable 3–manifold is said to be essential if it has the following five properties:*

1. *S is bicollared;*
2. *the inclusion homomorphism $\pi_1(S_i) \to \pi_1(M)$ is injective for every component S_i of S;*
3. *no component of S is a 2–sphere;*
4. *no component of S is boundary parallel;*
5. *S is nonempty.*

1.2.8. Stabilisers of the action. For the material in this subsection, please refer to [**42**]. If S is an orientable (not necessarily connected) surface in a connected orientable 3–manifold M, we can define the dual graph $\mathcal{G}_S$ of S. The vertices of $\mathcal{G}_S$ are in bijective correspondence to the components M_i of $M - S$, and the edges with the components S_i of S. A vertex v is incident to an edge e if and only if the corresponding component of S is contained in the closure of the component of $M - S$ corresponding to v.

There are a retraction map $r : M \to \mathcal{G}_S$ and an inclusion map $i : \mathcal{G}_S \to M$ such that $i \circ r$ is homotopy equivalent to the identity of $\mathcal{G}_S$. So $\pi_1(\mathcal{G}_S)$ is isomorphic to a subgroup and a quotient of $\pi_1(M)$. Thus, if we consider the universal cover $(\tilde{M}, p)$ of M, and let $\tilde{S} := p^{-1}(S)$, then the dual graph $\mathcal{T}_S$ of $\tilde{S}$ is a tree.

The action of $\pi_1(M)$ on $\tilde{M}$ gives rise to a simplicial action on $\mathcal{T}_S$, and since all manifolds involved are orientable, this action is without inversions, and the quotient of $\mathcal{T}_S$ by the action is $\mathcal{G}_S$. The stabiliser of a vertex of $\mathcal{T}_S$ is conjugate to $\text{im}(\pi_1(M_i) \to \pi_1(M))$ for some component M_i of $M - S$, and the stabiliser of an edge is conjugate to $\text{im}(\pi_1(S_i) \to \pi_1(M))$ for some component S_i of S.

In the above construction of an associated surface S, one can show that the map $\tilde{M} \to \mathcal{T}_v$ factors through a $\pi_1(M)$–equivariant map $\mathcal{T}_S \to \mathcal{T}_v$, which implies that the vertex and edge stabilisers of the action on $\mathcal{T}_S$ are included in the respective stabilisers of the action on $\mathcal{T}_v$. The inclusion is generally not an equality.

1.2.9. Surface detected by ideal point. In this subsection, we describe associated surfaces satisfying certain "non–triviality" conditions. Any essential surface gives rise to a graph of groups decomposition of $\pi_1(M)$, which shall be denoted by $\langle M_i, S_j, t_k \rangle$, where M_i are the components of $M - S$, S_j are the components of S, and t_k are generators of the fundamental group of the graph of groups arising from HNN–extensions.

Assume that $M - S$ consists of m components. For each component M_i of $M - S$ we fix a representative Γ_i of the conjugacy class of $\text{im}(\pi_1(M_i) \to \pi_1(M))$ as follows. Let $\mathcal{T}' \subset \mathcal{T}_S$ be a tree of representatives, i.e. a lift of a maximal tree in $\mathcal{G}_S$ to $\mathcal{T}_S$, and let $\{s_1, \ldots, s_m\}$ be the vertices of $\mathcal{T}'$, labelled such that s_i maps to M_i under the composite mapping $\mathcal{T}_S \to \mathcal{G}_S \to M$. Then let Γ_i be the stabiliser of s_i.

Any essential surface S which does not contain parallel copies of one of its components is called *detected by an ideal point* of the character variety with Serre tree $\mathcal{T}_v$ if

S1. every vertex stabiliser of the action on $\mathcal{T}_S$ is included in a vertex stabiliser of the action on $\mathcal{T}_v$,

S2. every edge stabiliser of the action on $\mathcal{T}_S$ is included in an edge stabiliser of the action on $\mathcal{T}_v$,

S3. if M_i and M_j, where $i \neq j$, are identified along a component of S, then there are elements $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ such that $\gamma_i\gamma_j$ acts as a loxodromic on $\mathcal{T}_v$,

S4. each of the generators t_i can be chosen to act as a loxodromic on $\mathcal{T}_v$.

LEMMA 1.12. *An essential surface in M detected by an ideal point ξ of a curve C in $\mathfrak{X}(M)$ is associated to the action of $\pi_1(M)$ on the Serre tree $\mathcal{T}_v$.*

PROOF. Denote the essential surface by S, and choose a sufficiently fine triangulation of M such that the 0–skeleton of the triangulation is disjoint from S, and such that the intersection of any edge in the triangulation with S consists of at most one point. Give $\tilde{M}$ the induced triangulation. There is a retraction $\tilde{M} \to \mathcal{T}_S$, which we may assume to be simplicial, and we now wish to define a map $\mathcal{T}_S \to \mathcal{T}_v$.

Note that the vertices $\{s_1, \ldots, s_m\}$ of the tree of representatives are a complete set of orbit representatives for the action of $\pi_1(M)$ on the 0–skeleton of $\mathcal{T}_S$. Condition S3 implies that we may choose vertices $\{v_1, \ldots, v_m\}$ of $\mathcal{T}_v$ such that v_i is stabilised by Γ_i, and if $M_i \neq M_j$, then $v_i \neq v_j$.

Define a map f^0 between the 0–skeleta of $\mathcal{T}_S$ and $\mathcal{T}_v$ as follows. Let $f^0(s_i) = v_i$. For each other vertex s of $\mathcal{T}_S$ there exists $\gamma \in \pi_1(M)$ such that $\gamma s_i = s$ for some i. Then let $f^0(s) = \gamma f^0(s^i)$. This construction is well–defined by the condition on the vertex stabilisers, and we therefore obtain a $\pi_1(M)$–equivariant map from

$\mathcal{T}_S^0 \to \mathcal{T}_v^0$. Moreover, this map extends uniquely to a map $f^1 : \mathcal{T}_S \to \mathcal{T}_v$, since the image of each edge is determined by the images of its endpoints. Since $v_i \neq v_j$ for $i \neq j$, and since each t_k acts as a loxodromic on $\mathcal{T}_v$, the image of each edge of $\mathcal{T}_S$ is a path of length greater or equal to one in $\mathcal{T}_v$.

If f^1 is not simplicial, then there is a subdivison of $\mathcal{T}_S$ giving a tree $\mathcal{T}_{S'}$ and a $\pi_1(M)$–equivariant, simplicial map $f : \mathcal{T}_{S'} \to \mathcal{T}_v$. There is a surface S' in M which is obtained from S by adding parallel copies of components such that $\mathcal{T}_{S'}$ is the dual tree of $\tilde{S}'$.

As before, choose a sufficiently fine triangulation of M such that the 0–skeleton of the triangulation is disjoint from S', and such that the intersection of any edge in the triangulation with S' consists of at most one point, and give $\tilde{M}$ the induced triangulation. The composite map $\tilde{M} \to \mathcal{T}_{S'} \to \mathcal{T}_v$ is $\pi_1(M)$–equivariant and simplicial, and the inverse image of midpoints of edges descends to the surface S' in M. Thus, S' is associated to the action of $\pi_1(M)$ on $\mathcal{T}_v$. ■

1.2.10. Limiting character. The *limiting character* at an ideal point ξ of a curve C in $\mathfrak{X}(M)$ is defined by the trace functions $I_\gamma(\xi) \in \mathbb{C} \cup \{\infty\}$ of Subsection 1.1.3. Let S be a surface associated to an ideal point of the character variety of a manifold M. Then the limiting character takes finite values when restricted to components of $M - S$, and hence corresponds to representations of these pieces. However, since we are at an ideal point of the character variety, these representations cannot be glued together to form a representation of the manifold. It follows from [**11**] that the limiting representation of any component of an associated surface is reducible, since it is shown that elements in the commutator group of an edge stabiliser have traces equal to two. This implies the following

LEMMA 1.13. [**42**] *The limiting representation of every component of an associated surface is reducible.*

1.2.11. Slopes and ideal points. Let M be a compact, orientable, irreducible 3–manifold, and T be a torus component of ∂M. The relationship between boundary slopes and sequences of representations is established in [**18**] and summarised in [**13**] as follows. A sequence $\{\rho_n\}$ of representations on a curve C in $\mathfrak{R}(M)$ is said to *blow up*, if there is an element $\gamma \in \pi_1(M)$ such that $I_\gamma(\rho_n) \to \infty$ as $n \to \infty$. Let S be an essential surface associated to the corresponding ideal point.

If there is an element γ in $\text{im}(\pi_1(T) \to \pi_1(M))$ such that $I_\gamma(\rho_n) \to \infty$, then up to inversion there is a unique element $h \in \text{im}(\pi_1(T) \to \pi_1(M))$ such that $\{I_h(\rho_n)\}$ is bounded. Then h is parallel to the boundary components of S on T.

If $\{I_h(\rho_n)\}$ is bounded for all $h \in \text{im}(\pi_1(T) \to \pi_1(M))$, then S may be chosen to be disjoint from T.

1.2.12. Roots of unity phenomenon. Let M be a compact, orientable, irreducible 3–manifold with boundary consisting of a single torus. Assume that there is an ideal point ξ of a curve C in $\mathfrak{X}(M)$ with the property that $I_\gamma(\xi) = \infty$ for some $\gamma \in \text{im}(\pi_1(\partial M) \to \pi_1(M))$. This ensures that an essential surface S which is associated to the action of $\pi_1(M)$ on $\mathcal{T}_v$ determined by $\mathcal{P}$ has non–empty boundary. The boundary components of S are a family of parallel simple closed curves on the torus ∂M, so they all lie in the same homotopy class. Let h be a representative of the particular homotopy class, and let $I_{\partial S}(\xi) = I_h(\xi)$. This is well–defined since the trace is invariant under conjugation and taking inverses.

A surface is called *reduced* if it has the minimal number of boundary components amongst all associated essential surfaces. Let $n(S)$ denote the greatest common divisor of the number of boundary components of the components of S.

THEOREM 1.14. [**11**] *Let M be a compact, orientable, irreducible 3–manifold with boundary consisting of a single torus, and let ξ be an ideal point of a curve C in $\mathfrak{X}(M)$ with the property that $I_\gamma(\xi) = \infty$ for some $\gamma \in \text{im}(\pi_1(\partial M) \to \pi_1(M))$.*

Let S be a surface associated to the action of $\pi_1(M)$ on $\mathcal{T}_v$ determined by $\mathcal{P}$. Then $I_{\partial S}(\xi) = \lambda + \lambda^{-1}$, where λ is a root of unity. Moreover, if S is reduced, then $\lambda^{n(S)} = 1$.

1.2.13. Associating surfaces. Given an essential surface S in a 3–manifold M, how can we decide whether S is detected by an ideal point of a curve in $\mathfrak{X}(M)$?

Denote the components of $M - S$ by $M_1, \ldots, M_k$ (where $M - S$ really stands for M minus an open collar neighbourhood of S). If S is detected by an ideal point, then the limiting character restricted to M_i is finite for all $i = 1, \ldots, k$. There is a natural map from $\mathfrak{X}(M)$ to the cartesian product $\mathfrak{X}(M_1) \times \ldots \times \mathfrak{X}(M_k)$, by restricting to the respective subgroups. Splittings along S which are detected by ideal points of curves in $\mathfrak{X}(M)$ correspond to points $(\chi_1, \ldots, \chi_k)$ in the cartesian product satisfying the following necessary conditions:

C1. $\chi_i \in \mathfrak{X}(M_i)$ is finite for each $i = 1, \ldots, k$.

C2. For each component of S, let $\varphi : S^+ \to S^-$ be the gluing map between its two copies arising from the splitting, and assume that $S^+ \subset \partial M_i$ and $S^- \subset \partial M_j$, where i and j are not necessarily distinct. Denote the homomorphism induced by φ on fundamental group by φ_*. Then for each $\gamma \in \text{im}(\pi_1(S^+) \to \pi_1(M_i))$, $\chi_i(\gamma) = \chi_j(\varphi_* \gamma)$.

C3. For each $i = 1, \ldots, k$, the restriction of χ_i to any component of S in ∂M_i is reducible.

C4. There is an ideal point ξ of a curve C in $\mathfrak{X}(M)$ and a connected open neighbourhood U of ξ on C such that the image of U under the map to the cartesian product contains an open neighbourhood of $(\chi_1, \ldots, \chi_k)$ on a curve in $\mathfrak{X}(M_1) \times \ldots \times \mathfrak{X}(M_k)$, but not $(\chi_1, \ldots, \chi_k)$ itself.

Note that C2 defines a subvariety of the cartesian product containing the image of $\mathfrak{X}(M)$ under the restriction map. According to Lemma 1.13, C3 is neccessary for a surface to be detected, and reduces the set of possible limiting characters to a subvariety. The last condition implies that at least one element of $\pi_1(M)$ has non–trivial translation length with respect to the action on Serre's tree, and the first condition implies that $\text{im}(\pi_1(M_i) \to \pi_1(M))$ is contained in a vertex stabiliser for each $i = 1, \ldots, k$.

Lemma 1.15. *Let S be a connected essential surface in a 3-manifold M. S is associated to an ideal point of the character variety of M if and only if there are points in the cartesian product of the character varieties of the components of $M - S$ satisfying conditions C1–C4.*

Proof. It is clear that the conditions are neccessary for any associated surface. We need to show that they are sufficient when S is connected and essential. Assume that S is non–separating. Let $A = \text{im}(\pi_1(M - S) \to \pi_1(M))$, and denote the subgroups of A corresponding to the two copies of S in $\partial(M - S)$ by A_1 and A_2. Since $\pi_1(M)$ is an HNN–extension of A across A_1 and A_2, there is $t \in \pi_1(M)$ such that $t^{-1}A_1 t = A_2$, and the subgroup generated by t and A is $\pi_1(M)$.

Let ξ be the ideal point provided by C4, and denote Serre's tree associated to ξ by $\mathcal{T}_v$. C1 implies that the subgroup A stabilises a vertex Λ of $\mathcal{T}_v$, and hence condition S1 is satisfied.

Note that A is finitely generated. C4 yields that the action of $\pi_1(M)$ on $\mathcal{T}_v$ is non–trivial, and Lemma 1.10 implies that either t is loxodromic with respect to the action on $\mathcal{T}_v$ or there is $a \in A$ such that ta or at is loxodromic. In the first case, we keep A_1 and A_2 as they are; in the second case, we replace t by ta and A_2 by $a^{-1}A_2 a$; and in the third case, we replace t by at and A_1 by aA_1a^{-1}. Thus, the generator t satisfies condition S4.

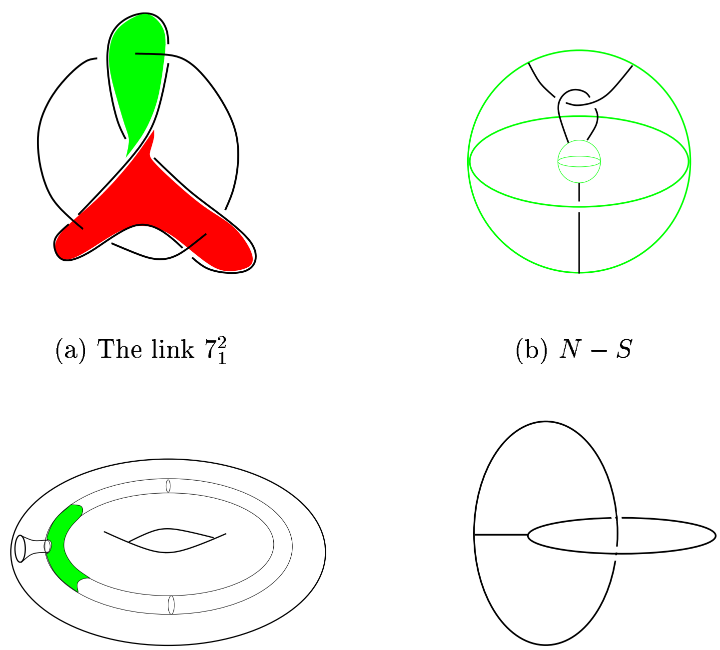
Since A stabilises Λ, $t^{-1}At$ stabilises $t^{-1}\Lambda$, and since t acts as a loxodromic, we have $t^{-1}\Lambda \neq \Lambda$. In particular, A_2 fixes these two distinct vertices, and hence the path $[\Lambda, t^{-1}\Lambda]$ pointwise, which implies that it is contained in an edge stabiliser. Thus, condition S2 is satisfied, and the lemma is proven in the case where S is connected, essential and non–separating, since condition S3 does not apply.

The proof for the separating case is similar, and will therefore be omitted. ■

Note that the conditions are not sufficient when S has more than one component, since condition C4 does not rule out the possibility that the limiting character is finite on all components of $M - S'$ for a proper subsurface S' of S.

1.3. The manifold $m137$

The manifold $m137$ in `SnapPea`'s census is one of the *Examples of non–trivial roots of unity at ideal points of hyperbolic 3–manifolds* in [**19**], where non–trivial means not equal to ± 1. We describe a reduced surface associated to an ideal point at which the non-trivial roots occur.



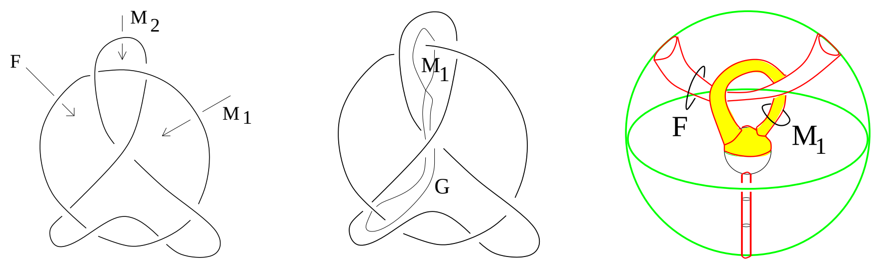
(a) The link 7_1^2 (b) $N - S$

(c) $T_1 \times I$ (d) Graph in S^3

FIGURE 1.1. The manifold $m137$

1.3.1. Let N be the manifold $m137$. We obtain N by 0–surgery on either component of the link 7_1^2 in S^3, which implies that N can be viewed as the complement of a knot in $S^2 \times S^1$. The following is a discussion of Figure 1.1. In (a), we see a thrice punctured disc in the link complement, and we assume that the 0–surgery is performed on its boundary curve. The union of the punctured disc and a meridian disc of the added solid torus is a thrice punctured sphere S in N. We may think of S as the intersection of $S^2 \times z$ with N in $S^2 \times S^1$. Cutting

N open along S results in the complement of three arcs in $S^2 \times I$, as shown in (b). The interior of $N - S$ is homeomorphic to the interior of an I–bundle over the once–punctured torus, the compact core of which is show in (c). A surface corresponding to one of the copies of S in the boundary of $N - S$ is shaded. The interior of $N - S$ is homeomorphic to the complement in S^3 of the trivalent graph of (d). For triangulations of and geometric structures on this space, see [**5, 27**] as well as Figure 1.3.



(a) $\pi_1(N)$ (b) $\pi_1(S)$ (c) $\pi_1(N - S)$

FIGURE 1.2. Generators for $m137$

1.3.2. Fundamental group. We compute a Wirtinger presentation for the fundamental group of the complement of the link 7_1^2 with meridians oriented according to Figure 1.2. Denote the peripheral system by $\{\mathcal{M}_1, \mathcal{L}_1\} \cup \{\mathcal{M}_2, \mathcal{L}_2\}$. The fundamental group is generated by $\mathcal{M}_1 = \mathfrak{a}$ and $\mathcal{M}_2 = \mathfrak{b}$, and we obtain a single relator which is equivalent to either of the relations $[\mathcal{M}_i, \mathcal{L}_i] = 1$, where the longitudes are words in the meridians:

$$\mathcal{L}_1 = \mathfrak{a}^2\mathfrak{b}^{-1}\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{a}\mathfrak{b}\mathfrak{a}^{1-}\mathfrak{b}^{-1}\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{b}^{-1},$$

$$\mathcal{L}_2 = \mathfrak{b}^2\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{a}\mathfrak{b}\mathfrak{a}\mathfrak{b}^{1-}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{a}\mathfrak{b}\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{a}^{-1}.$$

Note that $\mathcal{M}_1$ is homologous to $\mathcal{L}_2$ and that $\mathcal{M}_2$ is homologous to $\mathcal{L}_1$. If we perform 0–surgery on the second cusp of 7_1^2, we have

$$\pi_1(N) = \langle \mathcal{M}_1, \mathcal{M}_2 \mid \mathcal{L}_2 = 1 \rangle.$$

A peripheral system of N is given by $\{\mathcal{M}_1, \mathcal{L}_1\}$, where $\mathcal{M}_1$ is nullhomologous. We now give a HNN–extension of $\pi_1(N)$ which corresponds to the splitting along S. It can be observed from Figure 1.2, that the elements $\mathcal{M}_1$ and F correspond to generators of $\operatorname{im}(\pi_1(N-S) \to \pi_1(N))$, and if S_- and S_+ are the two copies of S in $\partial(N-S)$, then $\mathcal{M}_1 \in \operatorname{im}(\pi_1(S_-) \to \pi_1(N))$ implies $F \in \operatorname{im}(\pi_1(S_+) \to \pi_1(N))$. Let

$$F = \mathfrak{b}\mathfrak{a}\mathfrak{b}^{-1}, \qquad G = F^{-1}\mathfrak{a}^{-1}F\mathfrak{a}, \qquad H = F^{-1}\mathfrak{a}F\mathfrak{a}^{-1},$$
$$\text{and} \quad A = \langle \mathfrak{a}, F\rangle, \qquad A_1 = \langle \mathfrak{a}, G\rangle, \qquad A_2 = \langle F, H\rangle.$$

Each of the groups A, A_1 and A_2 is free in two generators, and $\pi_1(N)$ is an HNN–extension of A across A_1 and A_2 by $\mathcal{M}_2 = \mathfrak{b}$ with the relations

$$\mathfrak{b}\mathfrak{a}\mathfrak{b}^{-1} = F, \qquad \mathfrak{b}G\mathfrak{b}^{-1} = HF.$$

This HNN–extension corresponds to the splitting of N along S.

1.3.3. Representation spaces. With the method of Subsection 1.1.8, irreducible representations of N can (up to conjugation and birational equivalence) be parameterised by:

$$\rho(\mathfrak{a}) = \begin{pmatrix} m & 0 \\ \frac{(1-m)(b^2(m^2-1)-m)}{b^2(m-1)+m^2+m^3} & m^{-1} \end{pmatrix}, \qquad \rho(F) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}$$
$$\rho(\mathfrak{b}) = \begin{pmatrix} \frac{b^2(1-m)+b^4(1-m)^2-m^2}{b(m-1)(b^2(m-1)+m^2+m^3)} & b \\ \frac{b(1-m)(b^2(m^2-1)-m)}{b^2(m-1)+m^2+m^3} & b(m^{-1}-m) \end{pmatrix},$$

subject to

$$\begin{aligned} 0 &= m^3(1+m+m^2) \\ &\quad + b^2m(m^2-1)(1-m+m^2+2m^4+m^5) - b^4(m-1)^2(1+m+m^2). \end{aligned}$$

Thus, as $b \to 0$, we observe that either $m \to 0$ or m tends to a non–trivial third root of unity, which we denote by ζ_3. Since

$$\operatorname{tr}\rho(\mathfrak{b}) = \frac{-m^4 - b^2m^3(1-m)^2 + b^4(1-m)^2}{bm(m-1)(b^2(m-1)+m^2+m^3)},$$

it follows that as $b \to 0$ and $m \to \zeta_3$, we have $\operatorname{tr}\rho(\mathfrak{b}) \to \infty$, and an ideal point ξ of the character variety is approached. Moreover, the A–polynomial can be computed from the above, and we obtain:

$$\begin{aligned} 0 = (m^{13} - l^4)(1-m)(1+m+m^2)^3 \\ - l^2m^3(1+m^2)(1+3m+m^2-2m^3-3m^4-2m^5+2m^6 \\ - 2m^7 - 3m^8 - 2m^9 + m^{10} + 3m^{11} + m^{12}). \end{aligned}$$

As $m \to \zeta_3$, we have $l \to 0$, and hence $\operatorname{tr}\rho(\mathcal{L}_1) \to \infty$. Thus, $\mathcal{M}_1$ is a strongly detected boundary slope associated to ξ. The limiting representations of the images of A, A_1 and A_2 in $\pi_1(N)$ are determined by:

$$\begin{array}{llll} \operatorname{tr}\rho(\mathfrak{a}) \to -1, & \operatorname{tr}\rho(F) \to -1, & \operatorname{tr}\rho(F\mathfrak{a}) \to -\zeta_3, & \operatorname{tr}\rho[\mathfrak{a}, F] \to -1, \\ \operatorname{tr}\rho(\mathfrak{a}) \to -1, & \operatorname{tr}\rho(G) \to -1, & \operatorname{tr}\rho(G\mathfrak{a}^{-1}) \to -1, & \operatorname{tr}\rho[\mathfrak{a}, G] \to 2, \\ \operatorname{tr}\rho(F) \to -1, & \operatorname{tr}\rho(H) \to -1, & \operatorname{tr}\rho(FH) \to -1, & \operatorname{tr}\rho[F, H] \to 2. \end{array}$$

Thus, the limiting representation is irreducible on A, and reducible on A_1 and A_2. Since both $\operatorname{tr}\rho(\mathfrak{b})$ and $\operatorname{tr}\rho(\mathcal{L}_1)$ blow up, Lemma 1.15 implies that S is a reduced associated surface.

1.3.4. Preview: Mutation. The manifold N is obtained by identifying the two copies S_- and S_+ of S in $\partial(N-S)$ using a homeomorphism φ. If one changes the gluing map by an involution τ, one obtains a possibly different manifold N'. In Chapter 2, a map $\overline{\mu}$ between (subvarieties in) the character varieties of N and N' is defined, and the results of that chapter together with the above calculations imply that study of $\overline{\mu}$ would also lead to the conclusion that S is detected by an ideal point of the character variety of N.

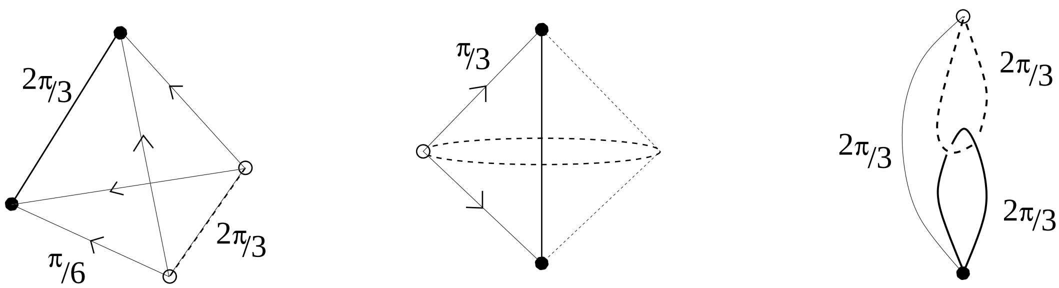


FIGURE 1.3. To obtain the limiting orbifold, first glue the faces meeting along one of the edges with cone angle $2\pi/3$ to obtain a spindle, and then identify the boundary discs of the spindle. The result is S^3 minus two points, with the labelled graph (minus its vertices) as the singular locus.

1.3.5. Preview: Degeneration. Since the limiting eigenvalue is a third root of unity, the limiting representation may correspond to a geometric decomposition of the orbifold $N(3,0)$. Using `SnapPea`, we can observe the deformation of the ideal triangulation of N as we approach $N(3,0)$. The manifold $m137$ has a triangulation with four tetrahedra, and as we approach $N(3,0)$ through positively oriented triangulations determined by the surgery coefficients $(p,0)$ with $p \geq 3$, three tetrahedra degenerate, whilst the remaining tetrahedron has limiting shape parameter $\frac{1}{2} + \frac{\sqrt{3}}{6}i$. In fact, $N(3,0)$ contains a Euclidean $(3,3,3)$–turnover whose complement is a hyperbolic 3–orbifold of volume approximately 0.68 which can be triangulated by a single ideal tetrahedron as shown in Figure 1.3. Note that the link of each vertex of the tetrahedron is a Euclidean $(3,3,3)$–turnover. Moreover, with the techniques of Chapter 4, we can associate a unique normal surface to the degeneration, and this normal surface turns out to be S.

CHAPTER 2

Mutation of 3–manifolds

The following is an application of Culler–Shalen theory to the study of mutative 3–manifolds. It can be used to compare the character varieties of mutative 3–manifolds, but it may also be employed to associate mutation surfaces to ideal points of the character variety.

Some of the arguments used in the case where the mutation surface is separating have been established by Cooper and Long in [**13**]. Here, one can find the construction of representations of the mutative manifold from representations of the original manifold, and the relationship between the A–polynomials of Conway mutants. In the following, their analysis is extended to Ruberman's symmetric surfaces in both the separating and non–separating case.

2.1. Definitions and statement of results

In the current section, some topological properties of mutative 3–manifolds are summarised, notation is fixed and the main results are stated. The remaining sections are then devoted to the relevant properties of characters on the mutation surfaces, the case where the surface does not separate the manifold, and the case where it does separate.

2.1.1. Mutation surfaces. A *mutation surface* (S, τ) is an incompressible, ∂–incompressible surface S, which is not boundary parallel, and an orientation preserving involution τ of S. The manifold obtained by cutting M open along S and regluing via τ is called a (S, τ)*–mutant* of M, and denoted by M^τ. Two manifolds are called *mutative* or *mutants* of each other if they are related by a finite sequence of mutations along mutation surfaces (S_i, τ_i). Note that mutation is an equivalence relation of 3–manifolds.

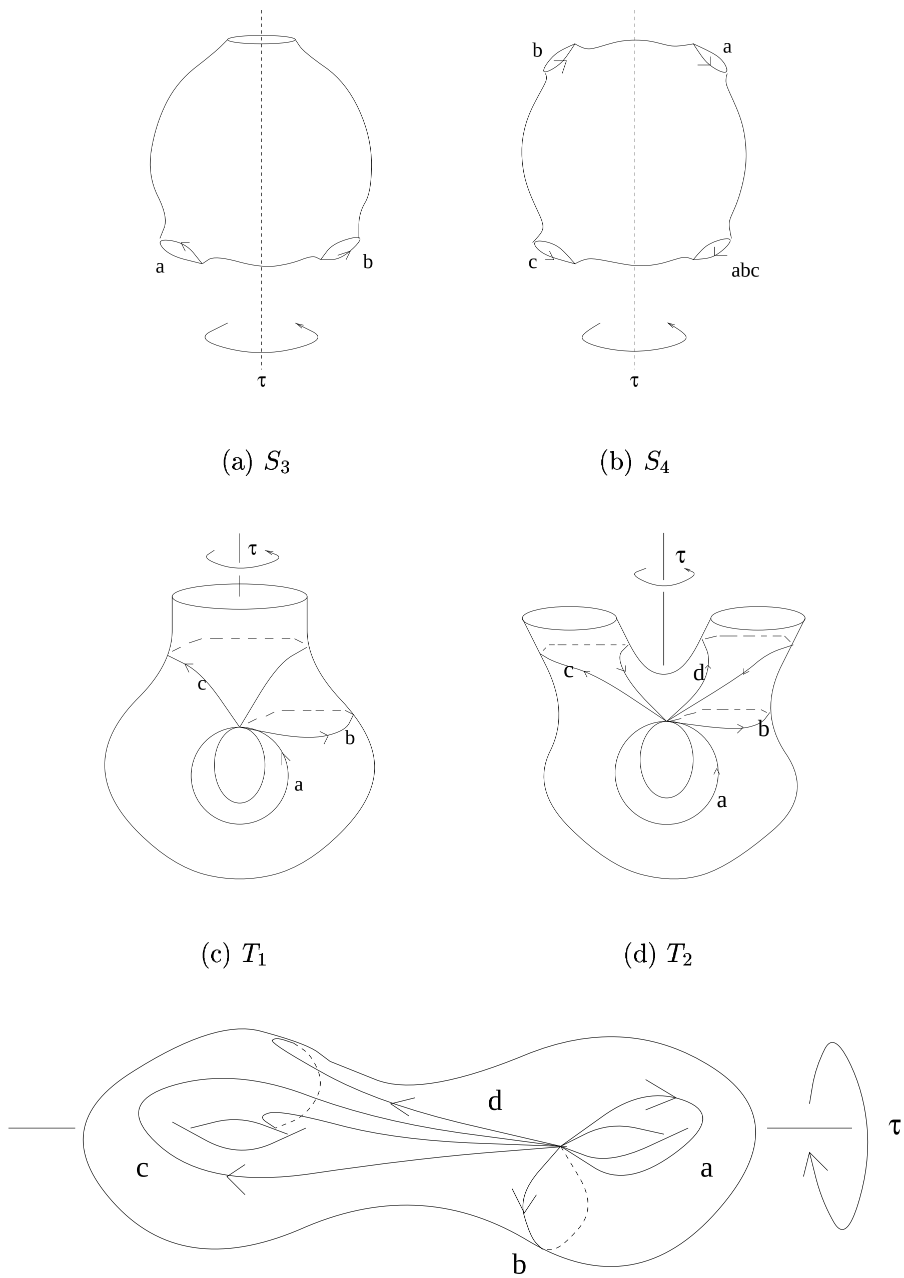
A special subclass amongst the mutation surfaces are the *symmetric surfaces* introduced by Ruberman in [**39**]. Here, S admits a hyperbolic structure and τ is an isometry of any hyperbolic structure on S. Moreover, τ is central in the mapping class group. In this case, Ruberman has shown that if M has a hyperbolic structure of finite volume, so does M^τ and $\mathrm{vol}(M) = \mathrm{vol}(M^\tau)$. A symmetric surface is one of the following: a thrice-punctured sphere S_3, a four-punctured sphere S_4, a once-punctured torus T_1, a twice-punctured torus T_2 or a genus-two surface G_2 along with suitable orientation preserving involutions τ as indicated in Figure 2.1. For geometric properties of mutation along thrice–punctured spheres see also [**1**].

2.1.2. Generalised Conway mutation. Classical mutation in knot complements as introduced by John Conway is accomplished by an embedded 2–sphere which intersects a knot in S^3 transversally in four points. It is noted in [**48**] that any Conway mutation can be realised by at most two mutations along genus two surfaces with the specified involution. This motivates the following terminology.

Two 3–manifolds are *generalised Conway mutants* of each other, if they are related by a sequence of mutations along separating symmetric surfaces S via involutions τ which induce the negative identity on first homology of S. The latter is satisfied by the involutions specified for T_1, T_2 and G_2. It may be possible to extend (S_3, τ) or (S_4, τ) to a mutation surface with the above properties.

The condition that the involution induces the negative identity on first homology also has nice consequences when the surfaces are non–separating. This condition shall be written as $\tau_\# = -E$ in the sequel.

2.1.3. Notation. A subvariety $\mathfrak{X}_\tau(M)$ in $\mathfrak{X}(M)$ will be defined, which consists of tentatively mutable characters, within which there is a subvariety $\mathfrak{F}(M)$ of S–reducible tentatively mutable characters, and the closure of $\mathfrak{X}_\tau(M) - \mathfrak{F}(M)$ in $\mathfrak{X}_\tau(M)$ is denoted by $\mathfrak{M}(M)$. The latter is a subvariety of $\mathfrak{X}^i(M)$, the closure of



(a) S_3 (b) S_4

(c) T_1 (d) T_2

(e) G_2

Figure 2.1. The symmetric surfaces and their involutions

the set of irreducible characters, and contains a dense set of mutable characters. The intersection of $\mathfrak{M}(M)$ with $\mathfrak{F}(M)$ is a subvariety $\mathfrak{F}^i(M)$ of irreducible S–reducible representations.[1] For example, if M has boundary consisting of a single torus, then $\mathfrak{X}_\tau(M) = \mathfrak{X}(M)$. If M is a 1–cusped complete hyperbolic 3–manifold of finite volume, then $\mathfrak{X}_0(M) \subseteq \mathfrak{M}(M)$.

All these notions descend to the $PSL_2(\mathbb{C})$–character variety $\overline{\mathfrak{X}}(M)$, and the objects are denoted by placing a bar over the previous notation.

2.1.4. Statement of results. We state the main results of this chapter using the notation introduced in the previous subsection.

PROPOSITION 2.1. *Let (S, τ) be a symmetric surface in a 3–manifold M, and let C be an irreducible component of $\overline{\mathfrak{M}}(M)$ which contains the character of a $PSL_2(\mathbb{C})$–representation such that the image of* $\mathrm{im}(\pi_1(S) \to \pi_1(M))$ *has trivial centraliser.*

Then C is birationally equivalent to an irreducible component of $\overline{\mathfrak{M}}(M^\tau)$.

The above proposition does not generally hold for the respective $SL_2(\mathbb{C})$–character varieties. Mutation of the figure eight knot complement along the fibre, for example, results in the manifold $m003$, and the smooth projective models of the $SL_2(\mathbb{C})$–character varieties are a torus and a sphere respectively (see Subsection 2.4.3).

PROPOSITION 2.2. *(Separating) Let (S, τ) be a separating symmetric surface in a 3–manifold M, and let C be an irreducible component of $\mathfrak{M}(M)$ which contains the character of an $SL_2(\mathbb{C})$–representation whose restriction to* $\mathrm{im}(\pi_1(S) \to \pi_1(M))$ *is irreducible.*

Then C is birationally equivalent to an irreducible component of $\mathfrak{M}(M^\tau)$.

The statements of the above propositions have the following consequences:

[1] The notation $\mathfrak{F}(M)$ was introduced in [**48**] because surfaces were denoted by F. Note that the variety $\mathfrak{F}(M)$ of [**48**] is now denoted by $\mathfrak{F}^i(M)$.

COROLLARY 2.3. *(Hyperbolic knots) Let $\mathfrak{k}$ be a hyperbolic knot and $\mathfrak{k}^\tau$ be a Conway mutant of $\mathfrak{k}$. Then $\overline{\mathfrak{X}}_0(\mathfrak{k})$ and $\overline{\mathfrak{X}}_0(\mathfrak{k}^\tau)$, as well as $\mathfrak{X}_0(\mathfrak{k})$ and $\mathfrak{X}_0(\mathfrak{k}^\tau)$, are birationally equivalent. Moreover, the associated factors of the A–polynomials are equal.*

According to the remarks made in Subsection 2.1.2, the previous corollary is a special case of the following:

COROLLARY 2.4. *(Separating in Hyperbolic) Let (S, τ) be a separating symmetric surface in a finite volume hyperbolic 3–manifold M.*

If S is a once–punctured torus, a twice–punctured torus or a genus two surface, i.e. if M and M^τ are generalised Conway mutants, then $\overline{\mathfrak{X}}_0(M)$ and $\overline{\mathfrak{X}}_0(M^\tau)$ are birationally equivalent, and $\mathfrak{X}_0(M)$ and $\mathfrak{X}_0(M^\tau)$ are birationally equivalent.

The restriction to the three surfaces in the above corollary is necessary in general. Firstly, there are no separating incompressible and ∂–incompressible thrice punctured spheres in hyperbolic 3–manifolds. Secondly, it is easy to construct examples of Conway mutation on 2–component links which cannot be extended to one of the other surfaces, and such that only a curve in each of the respective 2–dimensional Dehn surgery components is tentatively mutable.

For non–separating surfaces, one can similarly construct examples such that mutation along a twice–punctured torus or a thrice punctured sphere does not allow a general statement, which limits us to the following:

COROLLARY 2.5. *(Non–separating in Hyperbolic) Let (S, τ) be a non–separating symmetric surface in a finite volume hyperbolic 3–manifold M.*

If S is a once–punctured torus or a genus two surface, then $\overline{\mathfrak{X}}_0(M)$ and $\overline{\mathfrak{X}}_0(M^\tau)$ are birationally equivalent.

2.2. Tentatively mutable representations

Given a mutation surface (S, τ) in a 3–manifold M, the Seifert–Van Kampen theorem yields a decomposition of $\pi_1(M)$ with respect to $\pi_1(S)$, which allows to write down a presentation of $\pi_1(M^\tau)$ directly. This will be used to construct representations of M^τ from those of M, giving a map $\rho \to \rho^\tau$ with domain a certain subset of $\mathfrak{R}(M)$. The behaviour on $\pi_1(S)$ is crucial for such a construction. This motivates the following terminology.

2.2.1. Tentatively mutable in $\mathbf{SL_2}(\mathbb{C})$. Given a symmetric surface (S, τ), define a subvariety in $\mathfrak{R}(S)$, which consists of representations whose characters are invariant under τ by

$$\mathfrak{R}_\tau(S) = \{\rho \in \mathfrak{R}(S) \mid \operatorname{tr}\rho(\gamma) = \operatorname{tr}\rho(\tau_*\gamma) \quad \text{for all} \quad \gamma \in \pi_1(S)\}.$$

This subvariety descends to the character variety, since it is defined in terms of traces, and we write $\mathfrak{X}_\tau(S) = \mathrm{t}(\mathfrak{R}_\tau(S))$. In fact, τ induces a polynomial automorphism of $\mathfrak{X}(S)$, and $\mathfrak{X}_\tau(S)$ denotes the set of its fixed points.

Note that $\pi_1(S)$ is finitely generated, and recall that a character is uniquely determined by a point in $\mathbb{C}^m$ for some m which depends on the number of generators in a presentation of the fundamental group. $\mathfrak{R}_\tau(S)$ is therefore obtained from $\mathfrak{R}(S)$ by extending the set of defining equations by finitely many polynomial equations stating that the coordinates of the respective points are equal.

LEMMA 2.6. *Let (S, τ) be a symmetric surface as described in Figure 2.1 and ρ be a representation of $\pi_1(S)$. If $S = T_1$ or $S = G_2$, then $\mathfrak{R}(S) = \mathfrak{R}_\tau(S)$. Otherwise the character of ρ is invariant under τ if and only if it satisfies the following equations*

- *if $S = S_3$, $\pi_1(S_3) = \langle a, b\rangle$ then $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$,*
- *if $S = S_4$, $\pi_1(S_4) = \langle a, b, c\rangle$ then $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$ and $\operatorname{tr}\rho(c) = \operatorname{tr}\rho(abc)$,*
- *if $S = T_2$, $\pi_1(T_2) = \langle a, b, c\rangle$ then $\operatorname{tr}\rho(c) = \operatorname{tr}\rho(c^{-1}[a, b])$.*

PROOF. We describe the action of the involutions in terms of the generators indicated in Figure 2.1. Note that we choose base points for fundamental groups as fixed points of τ. Since we are working in the character variety, this choice does not matter. In the following we will use the following well known trace identities which hold for any $A, B \in SL_2(\mathbb{C})$:

$$\operatorname{tr} A = \operatorname{tr} A^{-1}, \tag{2.1}$$

$$\operatorname{tr} A \operatorname{tr} B = \operatorname{tr} AB + \operatorname{tr} AB^{-1}. \tag{2.2}$$

The statement of the lemma has to be verified for all surfaces. We do this representatively for G_2.

The fundamental group of G_2 is defined by the four generators a, b, c, d, and the single relation $[a,b][c,d] = 1$. The involution τ is described as follows:

$$\tau(a) = a^{-1}, \quad \tau(b) = ab^{-1}a^{-1}, \quad \tau(c) = ab^{-1}a^{-1}c^{-1}b, \quad \tau(d) = b^{-1}d^{-1}aba^{-1}.$$

Recall from Subsection 1.1.2 that the character of a representation is parametrised by the point $(\operatorname{tr}\rho(f), \operatorname{tr}\rho(fg), \operatorname{tr}\rho(fgh)) \in \mathbb{C}^{14}$, where f, g, $h \in \{a,b,c,d\}$ and $f < g < h$ in a lexicographical ordering. Using the relation in the fundamental group, we have the following identities:

$$\begin{aligned}\tau(c) &= ab^{-1}a^{-1}c^{-1}b = (b^{-1}cd)c^{-1}(d^{-1}c^{-1}b),\\ \tau(d) &= b^{-1}d^{-1}aba^{-1} = (b^{-1}c)d^{-1}(c^{-1}b).\end{aligned}$$

Thus, τ sends each generator to a conjugate of its inverse. The images under ρ and $\rho\tau$ therefore have equal trace. We have to verify double and triple products. The desired results either follow directly or require the very same trick, which we illustrate with the following example:

$$\begin{aligned}\operatorname{tr}\rho\tau(ad) &= \operatorname{tr}\rho(a^{-1}(b^{-1}cd^{-1}c^{-1}b)) && \mid \text{by definition of } \tau\\ &= \operatorname{tr}\rho(a)\operatorname{tr}\rho(d) - \operatorname{tr}\rho(ab^{-1}cd^{-1}c^{-1}b) && \mid \text{by (2.1) and (2.2)}\\ &= \operatorname{tr}\rho(a)\operatorname{tr}\rho(d) - \operatorname{tr}\rho(ab^{-1}d^{-1}aba^{-1}) && \mid \text{by } [a,b][c,d]=1\\ &= \operatorname{tr}\rho(a)\operatorname{tr}\rho(d) - \operatorname{tr}\rho(d^{-1}a) && \mid \text{by 2.1}\\ &= \operatorname{tr}\rho(ad) && \mid \text{by 2.2.}\end{aligned}$$

■

2.2.2. Codimension in $\mathbf{SL_2}(\mathbb{C})$. Let $\rho \in \mathfrak{R}_\tau(S)$ be an irreducible representation. Then there exists an element $X \in SL_2(\mathbb{C})$, which is unique up to sign, such that $\rho = X^{-1}\rho\tau X$. This is the basic fact which will be used to construct a map between subsets of our representation varieties. Since this unique inner automorphism only exists a priori for irreducible representations, the following fact will be of importance.

LEMMA 2.7. *Let (S, τ) be a symmetric surface. The subvariety of reducible representations has codimension one in the variety $\mathfrak{R}_\tau(S)$ of tentatively mutable representations of S. Moreover, this property is preserved under* t.

PROOF. We have to verify the lemma for all specified surfaces and involutions. The arguments are along the same lines, let us therefore consider the twice punctured torus. We have $\pi_1(T_2) \cong \langle a, b, c\rangle$. The space of representations has therefore nine dimensions. The variety $\mathfrak{R}_\tau(T_2)$ is defined by the additional equation $\operatorname{tr}\rho(c) = \operatorname{tr}\rho(c^{-1}[a, b])$, and is hence eight dimensional. But the set of reducible representations has seven dimensions, since we need two equations to state that two of the generators have a common 1–dimensional subspace with the third, and reducible representations always satisfy the required trace equality. Passing to the character variety, we merely subtract the three dimensions taken by conjugation throughout the above. This proves the claim for the twice punctured torus. ∎

Note that if $\tau_\# = -E$, then $\mathfrak{Red}(S) \subset \mathfrak{R}_\tau(S)$. For the twice punctured torus, this gives $\mathfrak{Red}(T_2) \subset \mathfrak{R}_\tau(T_2) \subset \mathfrak{R}(T_2)$, each inclusion with codimension one.

2.2.3. Tentatively mutable characters. If (S, τ) is a symmetric surface in a 3–manifold M, we call a representation $\rho \in \mathfrak{R}(M)$ *tentatively mutable with respect to* (S, τ) if its character restricted to S is invariant under τ. According to Lemma 2.6, the set $\mathfrak{R}_\tau(M)$ of these representations is a subvariety of $\mathfrak{R}(M)$ since we merely state that traces of finitely many elements have to be equal.

Let $\mathrm{t}(\mathfrak{R}_\tau(M)) = \mathfrak{X}_\tau(M)$. The S–reducible characters form a closed set in $\mathfrak{X}_\tau(M)$, which we denote by $\mathfrak{F}(M)$. Finally, we denote the closure of $\mathfrak{X}_\tau(M) - \mathfrak{F}(M)$ in $\mathfrak{X}_\tau(M)$ by $\mathfrak{M}(M)$. Clearly, we have $\mathfrak{M}(M) \subseteq \mathfrak{X}^i(M)$.

In particular, if $S = T_1$ or $S = G_2$, we have $\mathfrak{X}_\tau(M) = \mathfrak{X}(M)$, so $\mathfrak{X}^r(M) \subseteq \mathfrak{F}(M)$. Furthermore, the same is true for M^τ. In general, it is not true that $\mathfrak{X}_\tau(M) = \mathfrak{X}(M)$ implies $\mathfrak{X}_\tau(M^\tau) = \mathfrak{X}(M^\tau)$.

Assume that M is a finite volume hyperbolic 3–manifold which is not closed, and consider a symmetric surface (S, τ) in M. According to Ruberman's proofs in Section 2 of [**39**], we may choose the lift ρ_0 of the discrete and faithful representation such that punctures which are interchanged by τ have images with trace $+2$. Since the conditions in Lemma 2.6 are imposed on generators corresponding to boundary curves interchanged by τ, we have $\rho_0 \in \mathfrak{R}_\tau(M)$. Furthermore, by its faithfulness, ρ_0 cannot be reducible on S unless the second commutator group of $\pi_1(S)$ is trivial. Hence χ_0 is an S–irreducible character in $\mathfrak{M}(M)$. It follows from Lemma 2.7 that if a subvariety containing χ_0 is contained in $\mathfrak{X}_\tau(M)$, then it is contained in $\mathfrak{M}(M)$.

If $S = T_1$ or $S = G_2$, then $\mathfrak{X}_0(M) \subseteq \mathfrak{M}(M)$. We now argue that $\mathfrak{M}(M)$ contains at least a curve containing χ_0 if S is one of T_2, S_3, S_4.

If the punctures of S interchanged by τ lie on the same end of M, then $\mathfrak{X}_0(M) \subseteq \mathfrak{M}(M)$. If $S = T_2$ or S_3, then there is only one remaining case: the two punctures interchanged by τ lie on different ends. Then $\dim \mathfrak{X}_0(M) \geq 2$, and the condition in Lemma 2.6 adds one equation to the defining equations of $\mathfrak{X}_0(M)$ when we compute the contribution to $\mathfrak{M}(M)$. The resulting variety is therefore (at least) a curve. So assume that $S = S_4$. If the punctures of S lie on three or four different ends of M, then a dimension argument as above shows that $\mathfrak{M}(M)$ contains at least a curve. If the punctures of S lie on two different ends, it is easy to see that at least one of the two trace equations becomes redundant, and again we are left with a curve.

2.2.4. Tentatively mutable in $\mathbf{PSL_2}(\mathbb{C})$. In a similar way to the above, given a symmetric surface (S, τ), we define a subvariety in $\overline{\mathfrak{R}}(S)$, which consists of representations whose characters are invariant under τ by

$$\overline{\mathfrak{R}}_\tau(S) = \{\overline{\rho} \in \overline{\mathfrak{R}}(S) \mid \chi_{\overline{\rho}} = \chi_{\overline{\rho}\tau_*}\}.$$

This subvariety by definition descends to the character variety, and we write $\overline{\mathfrak{X}}_\tau(S) = \overline{\mathfrak{t}}(\overline{\mathfrak{R}}_\tau(S))$.

LEMMA 2.8. [2] *Let (S,τ) be a symmetric surface as described in Figure 2.1 and $\overline{\rho}$ be a $PSL_2(\mathbb{C})$–representation of $\pi_1(S)$. If S is one of the surfaces with boundary, then $\chi_{\overline{\rho}} = \chi_{\overline{\rho}\tau_*}$ if and only if there is a lift ρ of $\overline{\rho}$ such that $\chi_\rho = \chi_{\rho\tau_*}$.*

If $S = G_2$, then $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(G_2)$ either if $\overline{\rho}$ lifts to a $SL_2(\mathbb{C})$–representation, or if $(\operatorname{tr}\overline{\rho}(ad^{-1}))^2 = (\operatorname{tr}\overline{\rho}(bc^{-1}))^2 = (\operatorname{tr}\overline{\rho}(abd^{-1}))^2 = (\operatorname{tr}\overline{\rho}(b^{-1}cd))^2 = (\operatorname{tr}\overline{\rho}(acd))^2 = 0$.

PROOF. Let $\overline{\rho} \in \overline{\mathfrak{R}}(S)$, and assume that there is a lift ρ of $\overline{\rho}$ such that $\rho \in \mathfrak{R}_\tau(S)$. It then follows from Lemma 1.2 that $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, by choosing $\rho' = \rho\tau_*$ and $\epsilon = \mathrm{id}$.

Since the fundamental groups of the surfaces with boundary are free, every $PSL_2(\mathbb{C})$–representation of these surfaces lifts to a $SL_2(\mathbb{C})$-representation. We now verify the statement of the lemma for these surfaces.

- Case $S = T_1$. Since $\overline{\mathfrak{R}}(T_1) = \mathrm{q}(\mathfrak{R}(T_1))$ and $\mathfrak{R}(T_1) = \mathfrak{R}_\tau(T_1)$, there is nothing to prove. In particular, we have $\overline{\mathfrak{R}}(T_1) = \overline{\mathfrak{R}}_\tau(T_1)$.
- Case $S = T_2$. Let $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and $\rho \in \mathfrak{R}(S)$ be a lift of $\overline{\rho}$. By Lemma 1.2 we have $\chi_{\overline{\rho}} = \chi_{\overline{\rho}\tau_*}$ if and only if there is $\epsilon \in \operatorname{Hom}(\pi_1(S), \{\pm 1\})$ such that $\epsilon\chi_\rho = \chi_{\rho\tau_*}$. Now $\epsilon(a)\operatorname{tr}\rho(a) = \operatorname{tr}\rho\tau(a) = \operatorname{tr}\rho(a^{-1})$ forces $\epsilon(a) = 1$. Similarly, $\epsilon(b)\operatorname{tr}\rho(b) = \operatorname{tr}\rho\tau(b) = \operatorname{tr}\rho(ab^{-1}a^{-1})$ forces $\epsilon(b) = 1$. Then $\epsilon(bc)\operatorname{tr}\rho(bc) = \operatorname{tr}\rho\tau(bc) = \operatorname{tr}\rho(bc)$ yields $\epsilon(c) = \epsilon(b)$. Thus, $\epsilon = \mathrm{id}$, and the claim follows.
- Case $S = S_3$. Let $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and $\rho \in \mathfrak{R}(S)$ be a lift of $\overline{\rho}$. Since $\tau(a) = b$, we have $(\operatorname{tr}\overline{\rho}(a))^2 = (\operatorname{tr}\overline{\rho}(b))^2$. If $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$, then $\rho \in \mathfrak{R}_\tau(S)$. If $\operatorname{tr}\rho(a) = -\operatorname{tr}\rho(b)$, then define $\sigma(a) = \rho(a)$ and $\sigma(b) = -\rho(b)$. Then σ is a lift of $\overline{\rho}$ and $\sigma \in \mathfrak{R}_\tau(S)$. This completes the proof in this case.

[2] I thank Steven Boyer for pointing out that an earlier version of this lemma was incorrect.

- Case $S = S_4$. Let $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and $\rho \in \mathfrak{R}(S)$ be a lift of $\overline{\rho}$. We have $\chi_{\overline{\rho}} = \chi_{\overline{\rho}\tau_*}$ if and only if there is a homomorphism $\epsilon \in \mathrm{Hom}(\pi_1(S), \{\pm 1\})$ such that $\epsilon\chi_\rho = \chi_{\rho\tau_*}$. As above, considering the action of τ yields $\epsilon(a) = \epsilon(b) = \epsilon(c)$. If ϵ is trivial, then $\rho \in \mathfrak{R}_\tau(S)$. Otherwise, the character of the lift σ defined by $\sigma(a) = \rho(a)$, $\sigma(b) = \rho(b)$ and $\sigma(c) = -\rho(c)$ is invariant under τ.

We now have to consider $S = G_2$. It follows from Theorem 5.1 in [**22**] that $\overline{\mathfrak{X}}(G_2)$ has two topological components with the property that every representation in one of the components lifts to $SL_2(\mathbb{C})$, and every representation in the other component does not lift. We only have to consider the latter component since $\mathfrak{R}(G_2) = \mathfrak{R}_\tau(G_2)$.

Assume that $\overline{\rho}$ is a $PSL_2(\mathbb{C})$–representation of G_2 with representative matrices A, B, C, D for the generators a, b, c, d, such that $[A, B][C, D] = -E$. Then $\overline{\rho}$ does not lift to $SL_2(\mathbb{C})$. Now assume that $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$, and define a representation $\rho \in \mathfrak{R}(\mathfrak{F}_4)$ by $\rho(\alpha) = A$, $\rho(\beta) = B$, $\rho(\gamma) = C$ and $\rho(\delta) = D$. By assumption, there is $\epsilon \in \mathrm{Hom}(\mathfrak{F}_4, \{\pm 1\})$ such that $\epsilon\chi_\rho = \chi_{\rho\tau}$, where $\rho\tau$ is defined by

$$\rho\tau(\alpha) = A^{-1}, \qquad \rho\tau(\gamma) = (B^{-1}CD)C^{-1}(B^{-1}CD)^{-1},$$
$$\rho\tau(\beta) = AB^{-1}A^{-1}, \qquad \rho\tau(\delta) = (B^{-1}C)D^{-1}(B^{-1}C)^{-1}.$$

Then $\epsilon(a)\,\mathrm{tr}\,A = \epsilon(a)\,\mathrm{tr}\,\rho(\alpha) = \mathrm{tr}\,\rho\tau(\alpha) = \mathrm{tr}\,A^{-1}$ forces $\epsilon(a) = 1$. We similarly obtain $1 = \epsilon(b) = \epsilon(c) = \epsilon(d)$. But then $\epsilon = \mathrm{id}$, and we have

$$\begin{aligned}
\mathrm{tr}(AD) &= \epsilon(\alpha\delta)\,\mathrm{tr}\,\rho(\alpha\delta) = \mathrm{tr}\,\rho\tau(\alpha\delta) \\
&= \mathrm{tr}(A^{-1}(B^{-1}CD^{-1}C^{-1}B)) && \mid \text{by definition of } \rho\tau \\
&= \mathrm{tr}(A)\,\mathrm{tr}(D) - \mathrm{tr}(AB^{-1}CD^{-1}C^{-1}B) && \mid \text{by (2.1) and (2.2)} \\
&= \mathrm{tr}(A)\,\mathrm{tr}\,\rho(D) + \mathrm{tr}(AB^{-1}D^{-1}ABA^{-1}) && \mid \text{by } [A,B][C,D] = -E \\
&= \mathrm{tr}(A)\,\mathrm{tr}(D) + \mathrm{tr}(D^{-1}A) && \mid \text{by 2.1} \\
&= \mathrm{tr}(AD) + 2\,\mathrm{tr}(D^{-1}A) && \mid \text{by 2.2.}
\end{aligned}$$

Thus, $\operatorname{tr}(AD^{-1}) = 0$, and therefore $(\operatorname{tr}\overline{\rho}(ad^{-1}))^2 = 0$. The other stated trace identities follow similarly. This completes the proof of the lemma. We remark that there are representations satisfying the given equations. For example the assignment $\overline{\rho}(a) = \pm E$, $\overline{\rho}(b) = \pm E$,

$$\overline{\rho}(c) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \overline{\rho}(d) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

defines a representation of $\pi_1(G_2)$ into $PSL_2(\mathbb{C})$ which does not lift to $SL_2(\mathbb{C})$, and which itself is invariant under τ, i.e. $\overline{\rho} = \overline{\rho}\tau$. ■

2.2.5. Codimension in $\mathbf{PSL_2}(\mathbb{C})$. Let $\overline{\rho} \in \overline{\mathfrak{R}}_\tau(S)$ be an irreducible representation. Then there exists an element $X \in PSL_2(\mathbb{C})$, such that $\overline{\rho} = X^{-1}\overline{\rho}\tau X$. What is the centraliser of the image of an irreducible representation into $PSL_2(\mathbb{C})$?

The action of an element in $PSL_2(\mathbb{C})$ by conjugation is independent of the choice of lift to $SL_2(\mathbb{C})$. Thus, the centraliser of an element $\overline{A}$ in $PSL_2(\mathbb{C})$ consists of all elements $\overline{B}$ in $PSL_2(\mathbb{C})$ such that $\overline{B}A\overline{B}^{-1} = \pm A$ for a chosen lift $A \in SL_2(\mathbb{C})$ of $\overline{A}$. Thus, the centraliser of an element in $PSL_2(\mathbb{C})$ is the quotient of its centraliser in $SL_2(\mathbb{C})$ unless $\operatorname{tr} A = 0$. In case that $\operatorname{tr} A = 0$, the $PSL_2(\mathbb{C})$–centraliser of $\overline{A}$ contains the elements whose lifts to $SL_2(\mathbb{C})$ centralise or invert any lift of $\overline{A}$.

If a representation is irreducible, then there are two generators whose representative matrices have no common invariant subspace. One can try to describe irreducible $PSL_2(\mathbb{C})$–representations with non–trivial centraliser. As an example, let $A = \langle \alpha, \beta \rangle$, and let $\Gamma = \langle \alpha, \beta, k \mid k^{-1}\alpha k = \beta, k^{-1}\beta k = \alpha \rangle$. Consider the representation

$$\rho(\alpha) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \rho(\beta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of A. It extends to a unique $SL_2(\mathbb{C})$–representation of Γ, and the image of k is:

$$\rho(k) = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}.$$

The $SL_2(\mathbb{C})$–character of ρ is parameterised by $(0,0,0,0,\pm\sqrt{2},\pm\sqrt{2},0)$ with respect to the ordering $\alpha > \beta > k$ of the variables. Let us now consider representations of Γ into $PSL_2(\mathbb{C})$ which are equal to $\mathrm{q}(\rho)$ when restricted to A. We can change the image of k by elements in the $PSL_2(\mathbb{C})$–centraliser of $\mathrm{q}(\rho(A))$. For example the assignment

$$\rho'(k) = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix},$$

defines a $PSL_2(\mathbb{C})$–representation of Γ which has the same character on A, but clearly not on the whole group since the trace of $\rho'(k)$ is non–zero. Also note that this representation does not lift to $SL_2(\mathbb{C})$. Thus, the restriction map taking A–irreducible $SL_2(\mathbb{C})$–characters of Γ to characters of A is $1-1$, whilst for $PSL_2(\mathbb{C})$–characters it may be finite–to–one.

Let $\mathfrak{F}_2$ be the free group in two elements $\langle\alpha,\beta\rangle$. We have $\mathfrak{X}(\mathfrak{F}_2) \cong \mathbb{C}^3$, and the map $\overline{\mathfrak{X}}(\mathfrak{F}_2) \to \mathbb{C}^3$ given by $\chi_{\overline{\rho}} \to ((\operatorname{tr}\overline{\rho}(\alpha))^2, (\operatorname{tr}\overline{\rho}(\beta))^2, (\operatorname{tr}\overline{\rho}(\alpha\beta))^2)$ is a $2:1$ covering map.

LEMMA 2.9. *Let $\mathfrak{F}_2$ be the free group in two elements $\langle\alpha,\beta\rangle$, and consider the two–to–one parameterisation of the $PSL_2(\mathbb{C})$–character variety by the points $((\operatorname{tr}\overline{\rho}\alpha)^2, (\operatorname{tr}\overline{\rho}\beta)^2, (\operatorname{tr}\overline{\rho}\alpha\beta)^2)$ in $\mathbb{C}^3$. Then the set of irreducible representations with non–trivial centraliser is contained in the union of the three coordinate axes.*

PROOF. Assume that $\overline{\rho}$ is an irreducible representation of $\mathfrak{F}_2$ with non–trivial centraliser in $PSL_2(\mathbb{C})$. According to the above discussion of centralisers of irreducible representations, at least one of $(\operatorname{tr}\overline{\rho}\alpha)^2$ or $(\operatorname{tr}\overline{\rho}\beta)^2$ is equal to zero. Assume that $(\operatorname{tr}\overline{\rho}\alpha)^2 = 0$. Direct calculation shows that the centraliser of $\overline{\rho}(\mathfrak{F}_2)$ is non–trivial if and only if $(\operatorname{tr}\overline{\rho}\beta)^2 = 0$ or $(\operatorname{tr}\overline{\rho}\alpha\beta)^2 = 0$. If both are equal to zero, then the image of $\overline{\rho}$ is a Kleinian four group in $PSL_2(\mathbb{C})$ and equal to its centraliser, and if one of the traces is not equal to zero, then the centraliser has order equal to two. ■

LEMMA 2.10. *Let (S, τ) be a symmetric surface. The set of representations with non–trivial centralisers is contained in a finite union of subvarieties, each of which has codimension one in the variety $\overline{\mathfrak{R}}_\tau(S)$ of tentatively mutable $PSL_2(\mathbb{C})$–representations of S. Moreover, this property is preserved under $\overline{\mathfrak{t}}$.*

PROOF. The subvariety of reducible representations has codimension one since the proof of Lemma 2.7 applies again.

The set of irreducible representations with non–trivial centralisers are contained in a union of subvarieties each of which is defined by stating that two coordinates are equal to zero. Each of these subvarieties is easily observed to have codimension at least one in $\overline{\mathfrak{R}}_\tau(S)$ for each of the symmetric surfaces. ■

2.2.6. Extension lemma. Let A be a finitely generated group and $\varphi : A_1 \to A_2$ be an isomorphism between finitely generated subgroups of A. Denote by $\langle A, k, \varphi\rangle$ the HNN–extension of A by k across A_1 and A_2, where the subgroups are implicitly given by the isomorphism and vice versa. Define

$$\mathfrak{R}_\varphi(A) := \{\rho \in \mathfrak{R}(A) \mid \operatorname{tr}\rho(a) = \operatorname{tr}\rho\varphi(a) \quad \forall a \in A_1\}$$

to be the subvariety of representations of A which possibly extend to a representation of the above HNN–extension of A. Put $\mathfrak{t}(\mathfrak{R}_\varphi(A)) = \mathfrak{X}_\varphi(A)$.

Let $\rho \in \mathfrak{R}_\varphi(A)$ be a representation such that ρ restricted to A_1 is irreducible. Then restricted to A_1, ρ and $\rho\varphi$ are two irreducible representations with identical characters, and hence there is an element in $SL_2(\mathbb{C})$ which conjugates ρ to $\rho\varphi$. Moreover, since the centraliser of an irreducible representation in $SL_2(\mathbb{C})$ is $\langle -E\rangle$, the conjugating element is unique up to sign. Thus, the representation $\rho \in \mathfrak{R}_\varphi(A)$ extends to two representations of Γ which differ by the sign on the image of k. Moreover, note that there is an epimorphism $\Gamma \to \mathbb{Z}_2$ defined by sending each element of A to the identity, and k to the involution. Thus, if we descend to the $PSL_2(\mathbb{C})$–representation variety, the associated representation is unique.

We therefore formulate the above problem for projective representations. In the above construction we used the fact that the centraliser of an irreducible representation in $SL_2(\mathbb{C})$ is the center of $SL_2(\mathbb{C})$. Let

$$\overline{\mathfrak{R}}_\varphi(A) := \{\overline{\rho} \in \overline{\mathfrak{R}}(A) \mid \chi_{\overline{\rho}} \mid_{A_1} = \chi_{\overline{\rho}\varphi} \mid_{A_1}\},$$

and $\overline{\mathfrak{t}}(\overline{\mathfrak{R}}_\varphi(A)) = \overline{\mathfrak{X}}_\varphi(A)$. Then for any representation $\overline{\rho} \in \overline{\mathfrak{R}}_\varphi(A)$ such that $\overline{\rho} \mid_{A_1}$ is irreducible with trivial centraliser, there exists a unique representation $\overline{\rho}'$ of Γ such that $\overline{\rho}' \mid_A = \overline{\rho}$.

LEMMA 2.11. *(Extension lemma) Let* $\Gamma = \langle A, k \mid k^{-1}ak = \varphi(a) \quad \forall a \in A_1\rangle$, *and* $\overline{\mathfrak{X}}_\varphi(A)$ *as defined above. Let* V *be an irreducible component of* $\overline{\mathfrak{X}}(\Gamma)$. *Assume that* V *contains the character of a* $PSL_2(\mathbb{C})$*–representation which is irreducible on* A_1 *with trivial centraliser. Then the restriction map* $r : \overline{\mathfrak{X}}(G) \to \overline{\mathfrak{X}}_\varphi(A)$ *is a birational equivalence between* V *and* $\overline{r(V)}$.

PROOF. The restriction map is a polynomial map, and hence $W := \overline{r(V)}$ is an irreducible component of $\overline{\mathfrak{X}}_\varphi(A)$. If follows from Lemma 2.9 and Lemma 1.1, that the above construction of $PSL_2(\mathbb{C})$–representations of Γ from $PSL_2(\mathbb{C})$–representations of A is a well–defined 1–1 correspondence of $PSL_2(\mathbb{C})$–characters in V and W apart from a subvariety of codimension at least one. Thus, r has degree one and is therefore a birational isomorphism onto its image. ■

2.2.7. Mutable representations. Let (S, τ) be a symmetric surface. We have observed that if $\rho \in \mathfrak{R}_\tau(S)$ is irreducible, then there is a unique inner automorphism of $SL_2(\mathbb{C})$ realising the action of τ. Furthermore, X^2 centralises ρ and hence $X^2 = -E$, which implies that $\operatorname{tr} X = 0$.

We now determine the conditions under which there is an element in $SL_2(\mathbb{C})$ realising the action of τ for reducible $SL_2(\mathbb{C})$–representations. Assume that $\rho \in \mathfrak{R}_\tau(S)$ is reducible. Let $a \in \pi_1(S)$ and assume that $\tau(a) = b$, $\rho(a) = A$ and

$\rho(b) = B$. We wish to determine when the action of τ can be realised by an inner automorphism of $SL_2(\mathbb{C})$. Hence assume that $\pm A$ and $\pm B$ are non-trivial. If A and B commute, we may conjugate such that we get one of the normal forms (NF1) or (NF2) below, otherwise we may assume that A and B are given as in (NF3).

$$\text{(NF1)} \qquad A = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad b \neq 0$$

$$\text{(NF2)} \qquad A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \qquad B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \qquad b \in \{a, a^{-1}\}$$

$$\text{(NF3)} \qquad A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \qquad B = \begin{pmatrix} b & 1 \\ 0 & b^{-1} \end{pmatrix} \qquad b \in \{a, a^{-1}\}, a^2 \neq 1$$

If there exists $X \in SL_2(\mathbb{C})$ such that $X^{-1}AX = B$ and $X^{-1}BX = A$, then X is said to *simultaneously conjugate* A and B. By direct matrix computation one can show that there exists $X \in SL_2(\mathbb{C})$ which simultaneously conjugates the above pairs of matrices A and B if and only if

$$\text{(NF1(a))} \qquad B = A \qquad X = \pm \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \qquad X^2 = \begin{pmatrix} 1 & 2v \\ 0 & 1 \end{pmatrix} \qquad v \in \mathbb{C}$$

$$\text{(NF1(b))} \qquad B = A^{-1} \qquad X = \pm \begin{pmatrix} i & v \\ 0 & -i \end{pmatrix} \qquad X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad v \in \mathbb{C}$$

$$\text{(NF2(a))} \qquad B = A \qquad X = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \qquad X^2 = \begin{pmatrix} c^2 & 0 \\ 0 & c^{-2} \end{pmatrix} \qquad c \in \mathbb{C}^*$$

$$\text{(NF2(b))} \qquad B = A^{-1} \qquad X = \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \qquad X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad c \in \mathbb{C}^*$$

$$\text{(NF3)} \qquad b = a \qquad X = \pm i \begin{pmatrix} 1 & \frac{a}{a^2-1} \\ 0 & -1 \end{pmatrix} \qquad X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that if A is conjugate to C via T and there is an element X which simultaneously conjugates A and τA, then $T^{-1}XT$ simultaneously conjugates C and τC.

LEMMA 2.12. *Let $\rho \in \mathfrak{R}(S)$ be an upper triangular representation and assume there exists $X \in SL_2(\mathbb{C})$ such that $X^{-1}\rho X = \rho\tau$. Then we have the following cases:*

- *If $\rho(\pi_1(S))$ is abelian, then either $\rho(\tau\gamma) = \rho(\gamma)$ for all $\gamma \in \pi_1(S)$ or $\rho(\tau\gamma) = \rho(\gamma)^{-1}$ for all $\gamma \in \pi_1(S)$.*
- *If $\rho(\pi_1(S))$ is non-abelian and there exist an element $\delta \in \pi_1(S)$ such that the images of δ and $\tau(\delta)$ do not commute, then $\rho(\tau\gamma) = \rho(\gamma)^{-1}$ whenever $\rho(\gamma)$ is parabolic, and for all non–parabolic images, we have that the upper left entries of $\rho(\tau\gamma)$ and $\rho(\gamma)$ are equal and there exists a constant $c(\rho)$ such that if $\rho(\gamma) = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$ and $\rho(\tau\gamma) = \begin{pmatrix} x & z \\ 0 & x^{-1} \end{pmatrix}$, then $c(\rho) = \frac{y+z}{x-x^{-1}}$.*

  Moreover, X is unique up to sign.
- *If $\rho(\pi_1(S))$ is non-abelian and the images of γ and $\tau(\gamma)$ commute for all $\gamma \in \pi_1(S)$, then we have $\rho(\tau\gamma) = \rho(\gamma)$ for all non–parabolic images and we have either $\rho(\tau\gamma) = \rho(\gamma)$ for all parabolic images or $\rho(\tau\gamma) = \rho(\gamma)^{-1}$ for all parabolic images.*

Given $\rho \in \mathfrak{R}_\tau(M)$, the action of τ cannot be realised by an inner automorphism if and only if ρ is S–reducible and does not satisfy the conditions of the above lemma. A representation of M is accordingly called *mutable* or *non–mutable*. Note that the conjugating element may not be uniquely determined for S-reducble representations. If $\tau_\# = -E$, then any S–abelian representation $\rho \in \mathfrak{R}_\tau(M)$ is mutable and there is a one–complex–parameter family of elements realising the action of τ.

2.3. Non–separating surfaces

In this section, the statements of Subsection 2.1.4 which involve non–separating surfaces shall be proved.

2.3.1. Common double covers. If S is non–separating and τ induces the negative identity on $H_1(S)$, then M and M^τ have a common double cover since τ is central in the mapping class group. In particular, if M is hyperbolic, then M and M^τ are commensurable by definition. A first observation follows now by application of a result by D. D. Long and A. W. Reid for 1–cusped manifolds:

THEOREM 2.13. [**33**] *Suppose that $p : M \to T$ is a covering of 1–cusped orbifolds and suppose that the cusp of T is flexible. Then the induced map $p^* : \overline{\mathfrak{X}}_0(T) \to \overline{\mathfrak{X}}_0(M)$ is a birational equivalence.*

Since both the mutant and the double cover of a 1–cusped manifold are 1–cusped, it follows that the respective Dehn surgery components are birationally equivalent. Note that the above theorem uses a subgroup of finite index, whilst in the approach taken thus far subgroups of infinite index were utilised.

2.3.2. A natural map from $\overline{\mathfrak{M}}$(M) to $\overline{\mathfrak{M}}$(M$^\tau$). Let S be a non–separating symmetric surface in a 3–manifold M. We can describe $\pi_1(M)$ as an HNN–extension across $\pi_1(S)$ as follows. The boundary of $M - S$ contains two copies S_+ and S_- of S. Let $A = \text{im}(\pi_1(M-S) \to \pi_1(M))$, $A_1 = \text{im}(\pi_1(S_+) \to \pi_1(M)) \le A$ and $A_2 = \text{im}(\pi_1(S_-) \to \pi_1(M)) \le A$. Then there exists an element $k \in \pi_1(M) - A$ such that $\pi_1(M)$ is an extension of A by k across A_1 and A_2. We write this as:

$$\pi_1(M) = \langle A, k \mid k^{-1}A_1k = A_2\rangle.$$

The action of k is determined by the gluing map $S_+ \to S_-$. Performing mutation along S, we change the gluing map between the two copies of S in the boundary of $M - S$ by τ. We thus obtain a presentation of M^τ as follows:

$$\pi_1(M^\tau) = \langle A, k \mid k^{-1}\tau(A_1)k = A_2\rangle.$$

Let ρ be a representation of M. Note that $\rho(k)$ is only determined up to the centralizer of $\rho(A_1)$. Assume that ρ is tentatively mutable and $\rho(A_1)$ irreducible. $\rho(k)$ is now defined up to sign since the centralizer of an irreducible subgroup in $SL_2(\mathbb{C})$ is $\langle -E\rangle$. We write $(\rho|_A, \pm\rho(k))$ for such a pair of representations. Furthermore, $\rho(a)$ is conjugate to $\rho\tau(a)$ via some $X \in SL_2(\mathbb{C})$ for all $a \in A_1$. For the same reasons as above, X is defined up to sign. It follows that $\rho(k)^{-1}\rho(a_1)\rho(k) = \rho(a_2)$ is equivalent to $\rho(k)^{-1}X^{-1}\rho\tau(a_1)X\rho(k) = \rho(a_2)$. We can therefore define a representation $\rho^\tau \in R(M^\tau)$ by $\rho^\tau(a) := \rho(a)$ for all $a \in A$, and $\rho^\tau(k) = X\rho(k)$. This gives a pair $(\rho^\tau|_A, \pm\rho^\tau(k))$. We denote the corresponding map by $\overline{\mu}$, which is well defined up to sign, hence for projective representations. Since we can define an inverse map, we have a natural 1–1 correspondence of A_1–irreducible representations in $\mathrm{q}(\mathfrak{R}_\tau(M))$ and $\mathrm{q}(\mathfrak{R}_\tau(M^\tau))$, where $\mathrm{q} : \mathfrak{R}_\tau(M) \to \overline{\mathfrak{R}}_\tau(M)$ is the quotient map. This map is well–defined for equivalence classes of A_1–irreducible representations according to the remark made in Subsection 2.2.7 concerning elements which simultaneously conjugate. Hence it is well–defined for the corresponding characters in $\mathrm{q}(\mathfrak{M}(M))$ and $\mathrm{q}(\mathfrak{M}(M^\tau))$, where $\mathrm{q} : \mathfrak{M}(M) \to \overline{\mathfrak{M}}(M)$ is again the quotient map.

The map $\overline{\mu}$ can be extended in a similar fashion to $PSL_2(\mathbb{C})$–characters of $\overline{\mathfrak{R}}_\tau(M)$ which are S–irreducible, and whose image of $\pi_1(S)$ has trivial centraliser. We wish to show that $\overline{\mu}$ is a birational equivalence between suitable irreducible components, proving Proposition 2.1 for non–separating symmetric surfaces:

PROPOSITION 2.14. *Let (S,τ) be a* non–separating *symmetric surface in a 3–manifold M, and let C be an irreducible component of $\overline{\mathfrak{M}}(M)$ which contains the character of a $PSL_2(\mathbb{C})$–representation such that the image of $\pi_1(S)$ has trivial centraliser.*

Then C is birationally equivalent to an irreducible component of $\overline{\mathfrak{M}}(M^\tau)$.

PROOF. By definition, the restriction maps $r : \overline{\mathfrak{M}}(M) \to \overline{\mathfrak{X}}_\varphi(A)$ and $r^\tau : \overline{\mathfrak{M}}(M^\tau) \to \overline{\mathfrak{X}}_{\varphi\tau}(A)$ have range in a subvariety of $\overline{\mathfrak{X}}_\varphi(A) \cap \overline{\mathfrak{X}}_{\varphi\tau}(A)$. The construction of $\overline{\mu}$ gives $r(\chi) = r^\tau(\overline{\mu}\chi)$, whenever applicable. Since $\overline{\mu}$ is defined on a dense subset of C, the Extension Lemma 2.11 implies that it is the composition of the birational equivalences r and r^τ. This proves the proposition. ■

2.3.3. Proof of Corollary 2.5. Let M be a finite volume hyperbolic 3–manifold. Recall that, restricted to S, the complete representation $\overline{\rho}_0$ is irreducible and torsion free.

If $S = T_1$ or $S = G_2$, then $\mathfrak{X}_0(M) \subseteq \mathfrak{M}(M)$ and $\mathfrak{X}_0(M^\tau) \subseteq \mathfrak{M}(M^\tau)$ according to the discussion in Section 2.2.3. Since $\mathrm{q}(\mathfrak{X}_0) = \overline{\mathfrak{X}}_0$, it follows from Lemma 2.8 that $\overline{\mathfrak{X}}_0(M) \subseteq \overline{\mathfrak{M}}(M)$ and $\overline{\mathfrak{X}}_0(M^\tau) \subseteq \overline{\mathfrak{M}}(M^\tau)$. We can therefore apply Proposition 2.14 to the respective Dehn surgery components. It follows from [**39**] that $\overline{\mu}$ takes the complete representation of M to the complete representation of M^τ, and hence it restricts to a birational equivalence between the two Dehn surgery components. ■

2.4. The figure eight knot

The complement of the figure eight knot in S^3 is a once–punctured torus bundle with fibre a Seifert surface of the knot. Mutation along this surface results in the manifold $m003$.

2.4.1. Fundamental group. Let $\mathfrak{k}$ be the figure eight knot, $M = S^3 - \mathfrak{k}$ and $\Gamma = \pi_1(M)$. It is a known fact that a Seifert surface for $\mathfrak{k}$ is a once–punctured torus T_1. Such a surface is pictured in Figure 2.2. It consists of three "concentric" discs connected by bands. The involution τ of T_1 corresponds to a rotation by π about the axis perpendicular to the discs through their respective centres. We take the "uppermost" fixed point as a base point. $\pi_1(T_1)$ is free in two generators, and we choose them such that

$$\tau(a) = a^{-1}, \quad \tau(b) = ab^{-1}a^{-1} \quad \textit{and} \quad \tau([a,b]) = b^{-1}[a,b]b, \tag{2.3}$$

in order to meet our description of τ in Figure 2.1. Note that $[a, b]$ corresponds to a longitude of $\mathfrak{k}$. A presentation of Γ can be computed from Figure 2.2:

$$\Gamma = \langle t, a, b \mid t^{-1}at = aba, t^{-1}bt = ba\rangle,$$

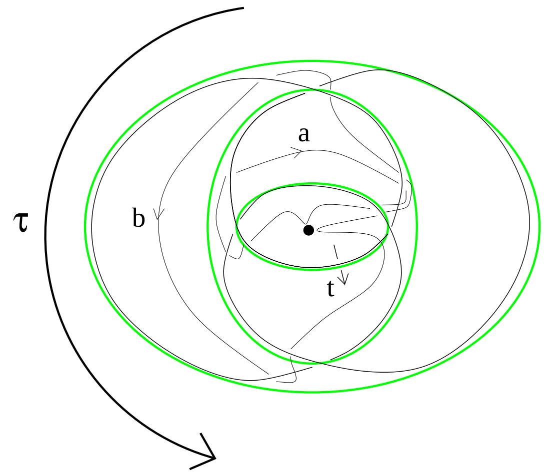


FIGURE 2.2. Mutation along the Seifert surface

where $\{t, [a, b]\}$ is a peripheral system. The action of t corresponds to the isomorphism Φ induced by the monodromy of the fibre bundle.

2.4.2. Mutation along the fibre. The isomorphism induced by the monodromy of the mutative manifold M^τ can be computed from τ and Φ:

$$\Phi^\tau(a) := \tau(\Phi(a)) = b^{-1}a^{-2}, \quad \Phi^\tau(b) := \tau(\Phi(b)) = ab^{-1}a^{-2}.$$

This gives

$$\Gamma^\tau = \pi_1(M^\tau) = \langle t, a, b \mid t^{-1}at = b^{-1}a^{-2}, t^{-1}bt = ab^{-1}a^{-2}\rangle.$$

Note that $H_1(M) \cong \mathbb{Z}$, while $H_1(M^\tau) \cong \mathbb{Z}_5 \oplus \mathbb{Z}$. A peripheral system of M^τ is given by $\{\mathcal{M}, \mathcal{L}\} = \{tb^{-1}, [a, b]\}$.

2.4.3. The character varieties. The presentations of both groups can be simplified to:

$$\Gamma = \langle t, a \mid t^{-1}a^{-1}t^{-1}ata^{-2}ta = 1\rangle, \tag{2.4}$$

$$\Gamma^\tau = \langle t, a \mid t^{-1}ata^2tat^{-1}a = 1\rangle. \tag{2.5}$$

It turns out that both character varieties have only one component which contains the character of an irreducible representation. Using the varieties $\mathfrak{C}(M)$ and $\mathfrak{C}(M^\tau)$ of Lemma 1.6, one obtains:

$$\begin{aligned} &\mathfrak{X}_0(M) = \{(x,y) \in \mathbb{C}^2 \mid 0 = 1 - y - y^2 + (-1+y)x^2 =: f\} \\ \text{and} \qquad &\mathfrak{X}_0(M^\tau) = \{(x,y) \in \mathbb{C}^2 \mid 0 = 1 + (-1+y)x^2 =: f^\tau\}, \end{aligned}$$

where $x = \operatorname{tr}\rho(t)$, $y = \operatorname{tr}\rho(a)$. The curve defined by f has no singularities and no singularities at infinity. The smooth projective completion $\tilde{\mathfrak{X}}_0(M)$ is therefore a torus. The curve defined by f^τ is rational, and a smooth projective model is therefore a sphere. This shows that the respective $SL_2(\mathbb{C})$–Dehn surgery components are not birationally equivalent.

To compute the components of the character varieties containing reducible representations, we use the following result:

PROPOSITION 2.15. [**18**] *Suppose that G is generated by g and h, and let ρ be a representation of G in $SL_2(\mathbb{C})$. Then ρ is reducible if and only if* $\operatorname{tr}\rho([g,h]) = 2$.

Using (2.4) and suitable trace identities, we obtain:

$$\operatorname{tr}\rho([t,a]) = \operatorname{tr}\rho(a^{-1}tat^{-1}a^{-1}) = \operatorname{tr}\rho(a)\operatorname{tr}\rho([t,a]) - \operatorname{tr}\rho(a).$$

Thus, $\operatorname{tr}\rho([t,a]) = 2$ if and only if $\operatorname{tr}\rho(a) = 2$. The curve containing the characters of reducible representations is therefore $\{y - 2 = 0\}$.

Similarly, using (2.5), we obtain:

$$\begin{aligned} \operatorname{tr}\rho([t,a]) &= \operatorname{tr}\rho(a^2t^{-1}ata^2) = \operatorname{tr}\rho(a^2)\operatorname{tr}\rho(t^{-1}ata^2) - \operatorname{tr}\rho(a) \\ &= \operatorname{tr}\rho(a^2)\operatorname{tr}\rho(a^{-1}ta^{-1}t^{-1}) - \operatorname{tr}\rho(a) \\ &= ((\operatorname{tr}\rho(a))^2 - 2)((\operatorname{tr}\rho(a))^2 - \operatorname{tr}\rho([t,a])) - \operatorname{tr}\rho(a), \end{aligned}$$

which gives $\operatorname{tr}\rho([t,a]) = 2$ if and only if $0 = (2-y)(1-y-y^2)$. The respective character varieties are therefore given by:

$$\mathfrak{X}(M) = \{(x,y) \in \mathbb{C}^2 \mid 0 = (2-y)(1-y-y^2+(-1+y)x^2)\}$$
$$\mathfrak{X}(M^\tau) = \{(x,y) \in \mathbb{C}^2 \mid 0 = (2-y)(1-y-y^2)(1+(-1+y)x^2)\}$$

2.4.4. $\mathbf{PSL_2}(\mathbb{C})$–character varieties. Note that there is a unique epimorphism from Γ onto $\mathbb{Z}_2$, and that the $\operatorname{Hom}(\Gamma, \mathbb{Z}_2)$–action corresponds to the involution $(x,y) \to (-x,y)$. The quotient map $\mathrm{q} : \mathfrak{X}(\Gamma) \to \overline{\mathfrak{X}}(\Gamma)$ can therefore be given by $\mathrm{q}(x,y) = (x^2, y)$. The same applies to Γ^τ. The $PSL_2(\mathbb{C})$–character varieties are therefore parameterised by:

$$\overline{\mathfrak{X}}(M) = \{(X,y) \in \mathbb{C}^2 \mid 0 = (2-y)(1-y-y^2+(-1+y)X)\}$$
$$\overline{\mathfrak{X}}(M^\tau) = \{(X,y) \in \mathbb{C}^2 \mid 0 = (2-y)(1-y-y^2)(1+(-1+y)X)\}$$

We will show that the rational maps between the Dehn surgery components induced by mutation are:

$$\overline{\mu} : \overline{\mathfrak{X}}_0(M) \to \overline{\mathfrak{X}}_0(M^\tau) \qquad (X,y) \to \left(\frac{1}{1-y}, y\right),$$
$$\overline{\mu}^{-1} : \overline{\mathfrak{X}}_0(M^\tau) \to \overline{\mathfrak{X}}_0(M) \qquad (X,y) \to \left(\frac{1-y-y^2}{1-y}, y\right).$$

2.4.5. Representations reducible on the fibre. Neither $\pi_1(M)$ nor $\pi_1(M^\tau)$ allow an irreducible representation which is non-abelian reducible on T_1, since $[a,b]$ is a longitude which commutes with t. It follows from (2.3) and Lemma 2.12 that all representations of M and M^τ are mutable.

By direct computation, we find that irreducible representations which are abelian on T_1 are parametrized by $1 = y + y^2$ in $\mathfrak{X}_0(M)$, and that there are no T_1–reducible irreducible representations in $\mathfrak{R}(M^\tau)$. Thus, there are only three points on each of the projective Dehn surgery components, on which the "mutation maps" $\overline{\mu}$ and $\overline{\mu}^{-1}$ are not defined a priori, and they correspond to the intersection with $\{(2-y)(1-y-y^2) = 0\}$.

2.4.6. Mutation map I. Irreducible representations of M are up to conjugacy given by:

$$\rho(t) = \pm \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \quad \text{and} \quad \rho(a) = \begin{pmatrix} u & 0 \\ \frac{(1-u)(m^2-u)}{mu} & u^{-1} \end{pmatrix},$$

where $m + m^{-1} = x$ and $u + u^{-1} = y$ are subject to:

$$0 = -m^2 + u + m^2u + m^4u - u^2 - 3m^2u^2 - m^4u^2 + u^3 + m^2u^3 + m^4u^3 - m^2u^4.$$

The representations degenerate to a reducible representation if and only if $u \to 1$, and irreducible representations are abelian on T_1 if and only if $m^2 = -1$.

In order to examine the map $\overline{\mu}$, we compute the conjugating element X and $\rho^\tau(t) = \rho(t)X$:

$$X = \begin{pmatrix} \frac{\sqrt{-m^2u+u^2}}{\sqrt{-1+m^2u}} & \frac{m(1+u)\sqrt{-m^2u+u^2}}{(m^2-u)\sqrt{-1+m^2u}} \\ \frac{(u-1)\sqrt{-m^2+u}}{m\sqrt{-u+m^2u^2}} & \frac{-\sqrt{-m^2u+u^2}}{\sqrt{-1+m^2u}} \end{pmatrix}, \qquad \operatorname{tr}(X) = 0,$$

$$\rho^\tau(t) = \begin{pmatrix} \frac{\sqrt{-m^2+u}(-1+u+m^2u)}{m\sqrt{-u+m^2u^2}} & \frac{(u+um^2)\sqrt{-m^2u+u^2}}{(m^2-u)\sqrt{-1+m^2u}} \\ \frac{(u-1)\sqrt{-m^2+u}}{m^2\sqrt{-u+m^2u^2}} & \frac{-\sqrt{-m^2u+u^2}}{m\sqrt{-1+m^2u}} \end{pmatrix}, \qquad \operatorname{tr}\rho^\tau(t) = \sqrt{\frac{1}{1-(u+u^{-1})}}.$$

Thus, the map $\overline{\mu}$ is as claimed the mapping $(X, y) \to (\frac{1}{1-y}, y)$. Moreover, as $u \to 1$ or $m^2 \to -1$, the entries of X stay finite and the limiting matrix realises the action of τ for the representations which are reducible on the fibre.

2.4.7. Mutation map II. Irreducible representations of M^τ are up to conjugacy given by:

$$\rho^\tau(t) = \pm \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \quad \text{and} \quad \rho^\tau(a) = \begin{pmatrix} u & 0 \\ \frac{(1-u)(m^2u-1)}{mu} & u^{-1} \end{pmatrix},$$

where $m + m^{-1} = x$ and $u + u^{-1} = y$ are subject to:

$$0 = 1 + 2m^2 + m^4 - u - m^2u - m^4u + u^2 + 2m^2u^2 + m^4u^2$$

The representations degenerate to a metabelian representation as $u \to 1$ and as $u \to m^{-2}$, which corresponds to the intersection with the curves $\{y = 2\}$ and $\{1 = y + y^2\}$. We obtain:

$$X = \begin{pmatrix} \frac{(-1+m^2u)}{\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} & \frac{m(1+u)}{\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} \\ \frac{(1-u)(-1+m^2u)}{m\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} & \frac{(1-m^2u)}{\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} \end{pmatrix},$$

$$\rho(t) = \begin{pmatrix} \frac{(1-m^2u)(-1-m^2+u)}{m\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} & \frac{1+m^2}{\sqrt{-u^2-m^4u^2+m^2(u+u^3}} \\ \frac{(1-u)(m^2u-1)}{m^2\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} & \frac{(1-m^2u)(1+u)}{m(1+u)\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} \end{pmatrix},$$

where $\quad \operatorname{tr}(X) = 0 \quad$ and $\quad \operatorname{tr}\rho(t) = \dfrac{(1-m^2u)(-1+m-m^2+u))}{m\sqrt{-u^2-m^4u^2+m^2(u+u^3)}} = \sqrt{\dfrac{1-y-y^2}{1-y}}.$

As above, this is the map we have described earlier, and X is well–defined for all representations in $\mathfrak{C}(\Gamma)$. It follows that the $PSL_2(\mathbb{C})$–Dehn surgery components of M and M^τ are homeomorphic.

2.4.8. Associating surfaces. The above calculations show that neither $\overline{\mathfrak{X}}_0(M)$ nor $\overline{\mathfrak{X}}_0(M^\tau)$ detects T_1. However, we have observed that all irreducible, T_1–irreducible representations of M and M^τ are abelian when restricted to T_1, and hence $\tau(\rho(\gamma)) = \rho(\gamma)^{-1}$ for these representations. In particular, they are mutable, and there is a 1–parameter family of elements in $PSL_2(\mathbb{C})$ which realise the action of τ.

Consider the following lift to $SL_2(\mathbb{C})$ of an irreducible $PSL_2(\mathbb{C})$–representation of M:

$$\rho(t) = \pm \begin{pmatrix} i & 1 \\ 0 & i^{-1} \end{pmatrix} \quad \text{and} \quad \rho(a) = \begin{pmatrix} u & 0 \\ i(u^{-1} - u) & u^{-1} \end{pmatrix},$$

subject to $0 = 1 + u + u^2 + u^3 + u^4$. These are dihedral representations, and they are abelian on the fibre. The elements realising the involution are given by:

$$X(z) = \pm \begin{pmatrix} iz & z \\ z - z^{-1} & i^{-1}z \end{pmatrix}$$

Thus, we obtain the following representations $\rho_z \in \mathfrak{R}(M^\tau)$:

$$\rho_z(t) = X(z)\rho(t) = \pm \begin{pmatrix} -z & 0 \\ i(z - z^{-1}) & -z^{-1} \end{pmatrix} \quad \text{and} \quad \rho_z(a) = \rho(a).$$

One may verify that these representations are abelian. The construction gives us a map $\mathbb{C} \to \mathfrak{R}(M^\tau) \to \overline{\mathfrak{X}}(M^\tau)$, which is non–constant since $\operatorname{tr} \rho_z(t) = -(z + z^{-1})$, and the image is therefore a curve in $\overline{\mathfrak{X}}(M^\tau)$. At an ideal point of this curve, the conditions of Lemma 1.15 are satisfied with respect to the surface T_1. One can do a similar construction for the other points in $\overline{\mathfrak{X}}^i(M) \cap \overline{\mathfrak{F}}(M)$.

Going the other way, i.e. using characters in $\overline{\mathfrak{X}}^i(M^\tau) \cap \overline{\mathfrak{F}}(M^\tau)$, one only obtains a curve in $\overline{\mathfrak{X}}(M)$ for the point corresponding to the intersection with $\{y = 2\}$, whilst the points in the intersection with $\{1 = y + y^2\}$ yield a map $\mathbb{C} \to \overline{\mathfrak{X}}(M)$ which is constant.

2.5. Separating surfaces

This section contains proofs of the statements of Subsection 2.1.4 which involve separating surfaces.

2.5.1. Reducible representations. If τ induces the negative identity on $H_1(S)$, then the character of any abelian representation is contained in $\mathfrak{X}_\tau(M)$. Recall that any reducible representation has the same character as some abelian representation. Hence, $\mathfrak{Red}(M) \subseteq \mathfrak{R}_\tau(M)$ and the closed set $\mathrm{t}(\mathfrak{Red}(M))$ of characters of reducible representations is carried by abelian representations and contained in $\mathfrak{X}_\tau(M)$. Using the Mayer–Vietoris exact sequence, one can show that $H_1(M) \cong H_1(M^\tau)$. This induces a natural isomorphism between the respective abelian representations and hence between the closed sets in $\mathfrak{X}(M)$ and $\mathfrak{X}(M^\tau)$ corresponding to reducible representations. This proves the following

PROPOSITION 2.16. *If M is a 3–manifold and (S, τ) a separating mutation surface such that τ induces the negative identity on first homology of S, then $\mathfrak{Red}(M) \subseteq \mathfrak{R}_\tau(M)$ and $\mathfrak{X}^r(M) \cong \mathfrak{X}^r(M^\tau)$.*

2.5.2. A natural map from $\mathfrak{M}$(M) to $\mathfrak{M}$($\mathbf{M}^\tau$). Given a symmetric surface (S, τ), there is a fixed point of τ. If we take this as the base point of the following fundamental groups, we get a decomposition

$$\pi_1(M) \cong \pi_1(M_-) \star_{\pi_1(S)} \pi_1(M_+).$$

$\mathfrak{R}(M)$ can be viewed as a subspace in $\mathfrak{R}(M_-) \times \mathfrak{R}(M_+)$, and the inclusion map is given by the restriction of ρ to the respective subgroups. If for a given $\rho_- \in \mathfrak{R}(M_-)$, there exists $\rho_+ \in \mathfrak{R}(M_+)$, such that they agree on $\pi_1(F)$, we say that ρ_- extends to a representation in $\mathfrak{R}(M)$. Similarly, $\mathfrak{R}(M^\tau) \subseteq \mathfrak{R}(M_-) \times \mathfrak{R}(M_+)$.

Let $\rho \in \mathfrak{R}_\tau(M)$ be an S–irreducible representation. We know that $\rho_-\tau$ is equivalent to ρ_- on $\pi_1(S)$, i.e. there is an element $X \in SL_2(\mathbb{C})$ such that $\rho_- = X^{-1}\rho_-\tau X$ on $\pi_1(F)$, and X is defined up to sign.

We can now define a representation ρ^τ of M^τ as follows: Let $\rho_+^\tau = \rho_+$ on $\pi_1(M_+)$ and $\rho_-^\tau = X^{-1}\rho_-\tau X$ on $\pi_1(M_-)$. $\rho^\tau = (\rho_-^\tau, \rho_+) \in \mathfrak{R}(M^\tau)$ is well defined, since both definitions agree on the amalgamating subgroup, and the map $\rho \to \rho^\tau$ only depends upon the inner automorphism induced by X. Note that both ρ and ρ^τ are irreducible and $\rho^\tau \in \mathfrak{R}_\tau(M^\tau)$.

LEMMA 2.17. *Let (S, τ) be a symmetric surface such that S is separating in M. Then there is a 1–1 correspondence of characters of S–irreducible representations in $\mathfrak{R}_\tau(M)$ and $\mathfrak{R}_\tau(M^\tau)$.*

PROOF. The above construction for a representation of the mutant manifold gives us a map between the respective representation spaces. We have to show that this map is well defined for equivalence classes of S–irreducible representations. Let $\rho = (\rho_-, \rho_+)$ and $\sigma = (\sigma_-, \sigma_+)$ be conjugate via $Y \in SL_2(\mathbb{C})$, and construct $\rho_-^\tau = X^{-1}\rho_-\tau X$ and $\sigma_-^\tau = Z^{-1}\sigma_-\tau Z$ as above. We need to show that ρ_-^τ is conjugate to σ_-^τ via Y. Note that $\rho_-\tau = Y\sigma_-\tau Y^{-1}$ by our assumption. Thus, restricted to $\pi_1(S)$, it follows that

$$X^{-1}Y\sigma_-\tau Y^{-1}X = X^{-1}\rho_-\tau X = \rho_- = Y\sigma_- Y^{-1} = YZ^{-1}\sigma_-\tau Z Y^{-1}.$$

Thus, $X^{-1}Y = YZ^{-1}$ modulo the centraliser of $\rho(\pi_1(F))$ in $SL_2(\mathbb{C})$. Since ρ is S–irreducible, we have $\mathbb{C}_{SL_2(\mathbb{C})}(\rho(\pi_1(F))) = \langle -E \rangle$. It follows that on $\pi_1(M_-)$

$$\rho_-^\tau = X^{-1}\rho_-\tau X = X^{-1}Y\sigma_-\tau Y^{-1}X = YZ^{-1}\sigma_-\tau ZY^{-1} = Y\sigma_-^\tau Y^{-1}.$$

Hence ρ^τ is conjugate to σ^τ via Y. This shows that the map is well defined on equivalence classes of S–irreducible representations. Furthermore, we can define an inverse map since $(M^\tau)^\tau = M$. This proves the claim. ∎

Note that throughout the above, we may interchange M_- and M_+. By the above lemma, we now have an isomorphism $\mu : \mathfrak{M}(M) \to \mathfrak{M}(M^\tau)$ defined everywhere apart from a subvariety $\mathfrak{F}^i(M)$ of characters of irreducible representations which are reducible on $\pi_1(S)$.

2.5.3. Proof of Proposition 2.2. If R is an irreducible component of $\mathfrak{R}_\tau(M)$ containing an S–irreducible representation, it follows from Lemma 2.7 that $\mathfrak{F}^i \cap \mathrm{t}(R)$ has codimension at least one in $\mathrm{t}(R) =: C$. Since the map μ is defined on an open, dense subset of C, the image $\mu(C) \subseteq \mathfrak{M}(M^\tau)$ is contained in a component C^τ of $\mathfrak{M}(M^\tau)$. The map μ is therefore an isomorphism between C and C^τ defined everywhere but on a codimesion one subvariety. In order to show that μ is a birational equivalence, it is now sufficient to show that it is rational.

Let K be the function field of R and let L be the function field of C. Consider the tautological representation $\mathcal{P} : \pi_1(M) \to SL_2(K)$. We have $\operatorname{tr} \mathcal{P}(\gamma) \in L \subseteq K$. Similarly, we have a tautological representation $\mathcal{P}^\tau$ on R^τ, which is the component in $\mathfrak{R}(M^\tau)$ corresponding to R under the map $\rho \to \rho^\tau$.

Let $\rho = (\rho_+, \rho_-) \in R$ and $\rho^\tau = (\rho_+, X^{-1}\rho_-\tau X) \in R^\tau$ be its image. We have $\mathcal{P} \mid_{M_+} = \mathcal{P}^\tau \mid_{M_+}$. By Proposition 1.1.1 of [**18**], we know that any representation equivalent to an irreducible representation on an irreducible component of $\mathfrak{R}(M)$ belongs itself to that component. Hence $X^{-1}\rho X \in R$ and $P \mid_{M_-}$ is defined by elements in $SL_2(K)$. This gives $\operatorname{tr} P^\tau(\gamma) \in L$ for all $\gamma \in \pi_1(M^\tau)$. The map μ is hence rational from C to C^τ. The existence of an inverse yields that the function fields L and L^τ are isomorphic and that C and C^τ are birationally equivalent. This completes the proof. Note that the above argument already shows that R and R^τ are birationally equivalent. ∎

2.5.4. Proof of Proposition 2.1. Proposition 2.1 for the case where S is separating follows since the above construction of representations also works for projective representations with trivial centraliser, and the argument in the above proof goes through if one uses Lemma 4.1 of [**6**] instead of Proposition 1.1.1 of [**18**]. ∎

2.5.5. Proof of Corollary 2.4. Assume that M is a finite volume hyperbolic 3–manifold, and that M and M^τ are generalised Conway mutants, i.e. S is separating and one of T_1, T_2, G_2.

If $\mathfrak{X}_0(M) \subseteq \mathfrak{M}(M)$, then $\overline{\mathfrak{X}}_0(M) \subseteq \overline{\mathfrak{M}}(M)$, since $\mathrm{q}(\mathfrak{X}_0) = \overline{\mathfrak{X}}_0$ and Lemma 2.8 applies. The two boundary components of any separating incompressible T_2 have to lie on the same boundary component of M, hence Lemma 2.6 implies for any of $S = T_1, T_2$ or G_2 that $\mathfrak{X}_\tau(M) = \mathfrak{X}(M)$ under our assumptions. Since ρ_0 is necessarily irreducible when restricted to S, we have $\mathfrak{X}_0(M) \subseteq \mathfrak{M}(M)$, and therefore $\overline{\mathfrak{X}}_0(M) \subseteq \overline{\mathfrak{M}}(M)$. Since ρ_0 is also torsion free, it follows that $\overline{\mathfrak{X}}_0(M)$ and $\mathfrak{X}_0(M)$ satisfy the assumptions of Propositions 2.1 and 2.2 respectively. It follows from [**39**] that the birational equivalence takes the complete representation of M to the complete representation of M^τ, and this proves the corollary. ∎

2.5.6. The above analysis together with Proposition 2.16 and the fact that a knot group abelianises to $\mathbb{Z}$ implies the following two results, which may be of interest:

PROPOSITION 2.18. *Let $\mathfrak{k}$ and $\mathfrak{k}^\tau$ be Conway mutants. If every component of $\mathfrak{X}(M)$ and $\mathfrak{X}(M^\tau)$ which contains the character of an irreducible representation contains the character of a representation which is irreducible on $\pi_1(S)$, then $\mathfrak{X}(M)$ and $\mathfrak{X}(M^\tau)$ are birationally equivalent.*

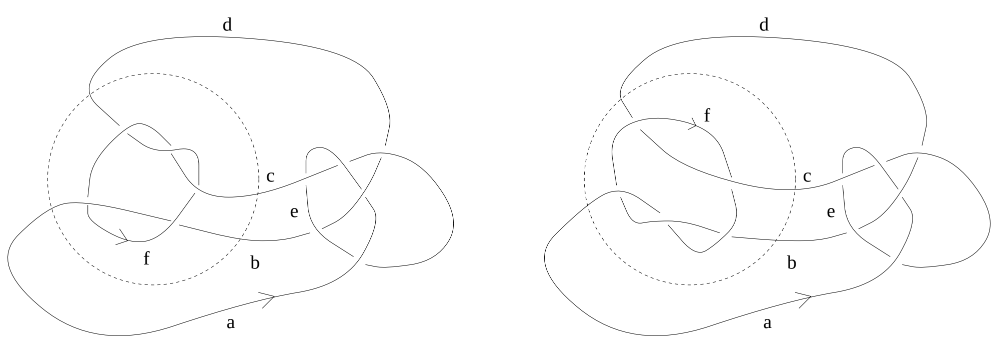
PROPOSITION 2.19. *Let M and M^τ be generalised Conway mutants. If every component of $\mathfrak{X}_\tau(M)$ and $\mathfrak{X}_\tau(M^\tau)$ which contains the character of an irreducible representation contains the character of a representation which is irreducible on $\pi_1(S)$, then $\mathfrak{X}_\tau(M)$ and $\mathfrak{X}_\tau(M^\tau)$ are birationally equivalent.*

Note that in the above proposition $\mathfrak{X}_\tau$ can be replaced by $\mathfrak{X}$ if M is a finite volume hyperbolic 3–manifold.

2.5.7. Conway mutation and the A–polynomial. We have observed that the peripheral subgroup of a knot is carried by a handlebody associated with the mutation. By the construction of the above map, the eigenvalue pairs of representations which restrict to irreducible representations on G_2 therefore do not change. Since the longitude is an element of the first commutator group, reducible representations contribute the factor $(l-1)$ to the A–polynomial. Thus:

PROPOSITION 2.20. [**13**] *Let $\mathfrak{k}^\tau$ be a Conway mutant of $\mathfrak{k}$. If $\mathfrak{F}^i(\mathfrak{k})$ is finite, then $A_{\mathfrak{k}}(l,m)$ is a factor of $A_{\mathfrak{k}^\tau}(l,m)$.*

2.6. The Kinoshita–Terasaka knot



(a) Kinoshita–Terasaka knot $\mathfrak{k}$ (b) Conway's mutant $\mathfrak{k}^\tau$

FIGURE 2.3. The Kinoshita-Terasaka knot and Conway's mutant

Let $\mathfrak{k}$ denote the Kinoshita–Terasaka knot and $\mathfrak{k}^\tau$ its Conway mutant as shown in Figure 2.3, and denote their complements in S^3 by M and M^τ respectively. Both knots have eleven crossings and trivial Alexander polynomial. Using `SnapPea`, one can verify that M and M^τ have hyperbolic volume approximately 11.219. Walter Neumann has used `Snap` to determine that the two complements are not commensurable.

THEOREM 2.21. [**48**] *$\mathfrak{X}(\mathfrak{k})$ and $\mathfrak{X}(\mathfrak{k}^\tau)$ are birationally equivalent. Moreover, the strongly detected boundary slopes of the Kinoshita–Terasaka knot and its mutant are identical.*

THEOREM 2.22. [**48**] *There are closed essential surfaces in the complement M of the Kinoshita–Terasaka knot which are detected by holes in the eigenvalue variety.*

2.6.1. Proof of Theorem 2.21. We wish to determine the set $\mathfrak{F}^i$ of characters corresponding to irreducible representations which are reducible on the four punctured sphere. This will be achieved by direct computation.

The fundamental groups of the inside M_- and the outside M_+ of S_4 can be computed from a Wirtinger presentation derived from the projection given in Figure 2.3.

$$\pi_1(M_-) = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{f} \mid r_1, r_2, r_3 \rangle \qquad \pi_1(M_+) = \langle a, b, c, d, e \mid r_1, r_2, r_3 \rangle$$

$$(r_1) \quad \mathfrak{b} = \mathfrak{f}\mathfrak{a}\mathfrak{f}^{-1} \qquad (r_4) \quad a = e^{-1}b^{-1}ecec^{-1}e^{-1}be$$

$$(r_2) \quad \mathfrak{c}\mathfrak{f}\mathfrak{c}^{-1}\mathfrak{f}^{-1}\mathfrak{c}^{-1}\mathfrak{a}\mathfrak{f}\mathfrak{a}^{-1} = 1 \qquad (r_5) \quad cece^{-1}c^{-1}ae^{-1}a^{-1} = 1,$$

$$(r_3) \quad \mathfrak{d} = \mathfrak{c}\mathfrak{b}\mathfrak{a}^{-1}, \qquad (r_6) \quad d = cba^{-1},$$

$$\pi_1(S_4) = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \rangle \qquad \pi_1(S_4) = \langle a, b, c \rangle.$$

Introducing the symbol f in order to give meridians linking number one with the respective knots, we obtain Wirtinger presentations of $\pi_1(M)$ and $\pi_1(M^\tau)$ by the following amalgamations:

$$\pi_1(M) = \langle \pi_1(M_-), \pi_1(M_+), f \mid \mathfrak{a} = a, \mathfrak{b} = b, \mathfrak{c} = c, \mathfrak{d} = d, \mathfrak{f} = f \rangle,$$

$$\pi_1(M^\tau) = \langle \pi_1(M_-), \pi_1(M_+), f \mid \mathfrak{a} = c^{-1}, \mathfrak{b} = d^{-1}, \mathfrak{c} = a^{-1}, \mathfrak{d} = b^{-1}, \mathfrak{f} = f^{-1} \rangle.$$

We give a brief overview of direct matrix computations, which we have done using `mathematica`. First, we compute all representations $\rho_- \in \mathfrak{R}(M_-)$ with $\operatorname{tr}\rho(\mathfrak{a}) = \operatorname{tr}\rho(\mathfrak{b}) = \operatorname{tr}\rho(\mathfrak{c}) = \operatorname{tr}\rho(\mathfrak{d}) = \operatorname{tr}\rho(\mathfrak{f})$ such that ρ_- is reducible on S_4. If the image of ρ_- is abelian on S_4, it follows that it is abelian on M_- and subsequently on M and M^τ.

Therefore assume that the image of ρ_- is reducible and non-abelian on S_4. Since we are only interested in the equivalence class of a representation, we may assume that $\rho_-(\pi_1(S_4))$ is generated by upper triangular matrices. It follows that $\operatorname{tr}\rho(\mathfrak{a}) \neq \pm 2$, and we can conjugate the representation such that it stays upper triangular and $\rho(\mathfrak{a})$ is diagonal while one of $\rho(\mathfrak{b})$ and $\rho(\mathfrak{c})$ has a non–negative upper right entry. These assumptions give $\rho(\mathfrak{a}) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$, $\rho(\mathfrak{c}) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix}$, $\rho(\mathfrak{f}) = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ as elements of $SL_2(\mathbb{C})$, which must satisfy the relations r_1 and r_2, such that the traces of all generators are equal, $\rho(\mathfrak{b})$ is upper triangular and the image of $\pi_1(S_4)$ is non–abelian.

This gives four representations ρ_1 to ρ_4. ρ_1 and ρ_3 have one parameter p and no relations, ρ_2 has three parameters p, u, x and one relation, ρ_4 has one parameter p and the relation $p^2 = -1$.

Using the identifications given in the presentations of $\pi_1(M)$ and $\pi_1(M^\tau)$, we now put $\rho(e) = \begin{pmatrix} l & m \\ n & o \end{pmatrix}$ and use the relations r_4 and r_5 to find out which of the above representations of M_- extend to representations ρ of M and M^τ respectively. It turns out that ρ_1, ρ_2 and ρ_3 extend to representations ρ of M and M^τ respectively, whereas ρ_4 does not. The representations which follow are all given in one parameter p, which is an eigenvalue of a meridian and specified as the zero of some polynomial in $\mathbb{C}[p]$. The parametrisations are given in Table 2.1.

$\rho \in$	$\rho \mid M_-$	parametrisation
$R(M)$	ρ_1	$F_1 = 1 - p^2 + 3p^4 - 4p^6 + 2p^8$
$R(M)$	ρ_2	$F_2 = 1 - p^4 + 3p^8 - 3p^{10} - p^{12} + 4p^{14} - 2p^{16} - p^{18} + p^{20}$
$R(M)$	ρ_3	$F_3 = 1 - 2p^2 + 3p^4 - 5p^6 + 2p^8$
$R(M^\tau)$	$\rho_1\tau$	$G_1 = 1 - p^2 + 3p^4 - 4p^6 + 2p^8$
$R(M^\tau)$	$\rho_2\tau$	$G_2 = 1 - p^4 + 3p^8 - 3p^{10} - p^{12} + 4p^{14} - 2p^{16} - p^{18} + p^{20}$
$R(M^\tau)$	$\rho_3\tau$	$G_3 = 1 - 2p^2 + 3p^4 - 5p^6 + 2p^8$
$R(M^\tau)$	$\rho_3\tau$	$G_4 = 1 - 3p^2 - p^4 + 3p^6 - p^8$

TABLE 2.1. Representations which are reducible on the Conway sphere

Note that $F_1 = G_1$, $F_2 = G_2$, $F_3 = G_3$ and that G_4 is not a factor of any of the above. Moreover, any two distinct polynomials from the above list have no zeros in common. Since the above give finite sets of points $\mathfrak{F}^i(\mathfrak{k})$ and $\mathfrak{F}^i(\mathfrak{k}^\tau)$ respectively, Theorem 2.21 follows from Propositions 2.19 and 2.20. ■

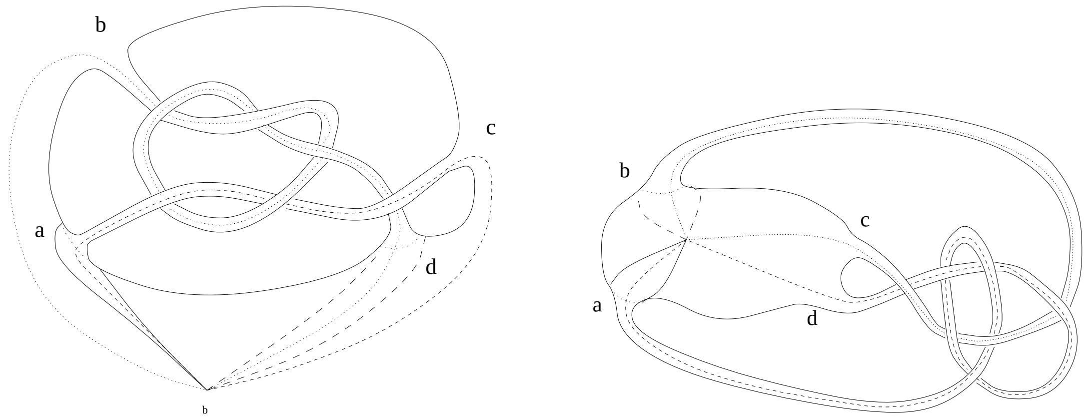
2.6.2. Proof of Theorem 2.22. Recall that there is a well defined eigenvalue map, taking a representation to a point in $\mathbb{C}^2$ by means of projection to the eigenvalues of meridian and longitude corresponding to a common invariant subspace. The closure of the image is a curve defined by a single polynomial in two variables. Let C be a component of $\mathfrak{R}(M)$. We call a pair (l, m) of eigenvalues a *hole* if the image of a connected open neighbourhood in C under the eigenvalue map contains a punctured neighbourhood of (l, m) but not (l, m) itself.

The relationship between boundary slopes and sequences of representations is explained in Subsection 1.2.11, and two different types of blow–ups are descibed. We observe a blow up of the second type either if there is a component in the character variety of dimension greater than one (so the inverse image of a point in the eigenvalue variety contains a whole curve) or if there is a hole in the eigenvalue variety. Examples for the first kind of behaviour haven been constructed in [**13**].

Consider the character of a representation ρ^τ of M^τ parametrised by a zero z of G_4. We choose an open neighbourhood U in an irreducible component of $\mathfrak{R}(M^\tau)$ containing ρ^τ, such that $U - \{\rho^\tau\}$ only contains representations which are F–irreducible. This is possible since it follows from [**17**] that the dimension of components of $\mathfrak{X}(M^\tau)$ is greater than zero. Let $({\rho_n}^\tau)$ be a sequence of representations in U such that $\lim_{n\to\infty} \mathrm{t}({\rho_n}^\tau) = \mathrm{t}(\rho^\tau)$. The map μ sends $U - \{\rho^\tau\}$ to some set in $\mathfrak{R}(M)$. This gives us a sequence of representations (ρ_n) in $\mathfrak{R}(M)$. We may assume that ${\rho_n}^\tau \mid_{M_-} = \rho_n \mid_{M_-}$, where M_- is a suitably chosen handlebody containing $\mathfrak{k}^\tau$. If the sequence (ρ_n) converges, it converges towards a representation which is reducible on $\pi_1(F)$, and the eigenvalue of a meridian is z or z^{-1}. But according to the calculation in the previous subsection, such a representation of M does not exist. Hence, the sequence (ρ_n) blows up, i.e. there is an element $g \in \pi_1(M)$ such that $\lim_{n\to\infty} \operatorname{tr} \rho_n(g) = \infty$. Since the eigenvalues of meridian and longitude are carried by M_-, they stay bounded and we have detected a closed essential surface in M.

Since the dimension of $\mathfrak{F}^i(M^\tau)$ and $\mathfrak{F}^i(M)$ is zero respectively, the components containing characters of representations corresponding to these points have dimension one by Lemma 2.7. Hence, the above argument shows that there are holes in the eigenvalue variety of M. This completes the proof of Theorem 2.22. ∎

2.6.3. Associated surfaces. Lemma 1.15 implies that the Conway sphere as well as any genus two surface obtained by joining boundary components of the sphere with annuli is a surface associated to the ideal points of $\mathfrak{X}(M)$ at which the holes in the eigenvalue variety occur. The two genus two surfaces and their involutions are shown in Figure 2.4.



(a) $\partial M'_-$ as seen from inside M'_-. τ acts as $(a,c)(b,d)$ modulo conjugation and inversion

(b) $\partial(M' - M'_-)$ as seen from inside M_+. τ acts as $(a,d)(b,c)$ modulo conjugation and inversion

FIGURE 2.4. Genus two surfaces realising the mutation

2.7. Remarks

2.7.1. Mutation surfaces and ideal points. The examples illustrate how the study of the maps μ and $\overline{\mu}$ can be used to show that mutation surfaces are associated to ideal points of the character variety.

2.7.2. Cutting along general surfaces. The propositions of Subsection 2.1.4 may be stated for arbitrary connected essential surfaces and suitable gluing maps, since their proofs only involve general properties of irreducible representations as well as HNN–extensions or amalgamated products. The definitions of $\overline{\mathfrak{X}}_\tau$ and $\mathfrak{X}_\tau$ merely require characters restricted to the surface to be invariant under the induced isomorphism on fundamental group, and if there is a character of a S–irreducible representation (with trivial centraliser) on some irreducible component V of $\overline{\mathfrak{M}}$ or $\mathfrak{M}$, then the set of characters on which the map $\overline{\mu}$ or μ is not a well–defined birational isomorphism is necessarily of codimension at least one.

2.7.3. Computing character varieties. A possible application of the Extension Lemma 2.11 could be the study of character varieties of manifolds via subvarieties in the character varieties of manifolds obtained by successively cutting along non–separating essential surfaces. For example in [**29**], this technique is used for the study of the character variety of the figure eight knot complement.

CHAPTER 3

The eigenvalue variety

To deal with multi–cusped manifolds, the A–polynomial of a 1–cusped manifold is generalised to an *eigenvalue variety* in Section 3.2. Boundary curves of essential surfaces arising at ideal points of the character variety are called *strongly detected*. The set of strongly detected boundary curves is determined in terms of Bergman's *logarithmic limit set*, which we describe in Section 3.1.

3.1. Bergman's logarithmic limit set

This section discusses the *logarithmic limit set* of [**3**], which we use to define the set of ideal points of algebraic varieties in $(\mathbb{C}-\{0\})^m$ in Sections 3.2 and 4.2. The logarithmic limit set describes the exponential behaviour of a variety at infinity.

3.1.1. Logarithmic limit set. Let V be a subvariety of $(\mathbb{C}-\{0\})^m$ defined by an ideal J, and let $\mathbb{C}[X^{\pm}] = \mathbb{C}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$. Bergman gives the following three descriptions of a *logarithmic limit set* in [**3**]:

1. Define $V_\infty^{(a)}$ as the set of limit points on S^{m-1} of the set of real m–tuples in the interior of B^m:

$$\left\{ \frac{(\log|x_1|, \dots, \log|x_m|)}{\sqrt{1+\sum(\log|x_i|)^2}} \mid x \in V \right\}. \tag{3.1}$$

2. Define $V_\infty^{(b)}$ as the set of m–tuples

$$(-v(X_1), \dots, -v(X_m)) \tag{3.2}$$

as v runs over all real–valued valuations on $\mathbb{C}[X^{\pm}]/J$ satisfying $\sum v(X_i)^2 = 1$.

3. Define the support $s(f)$ of an element $f \in \mathbb{C}[X^{\pm}]$ to be the set of points $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$ such that $X^\alpha = X_1^{\alpha_1} \cdots X_m^{\alpha_m}$ occurs with non–zero coefficient

in f. Then define $V_\infty^{(c)}$ to be the set of $\xi \in S^{m-1}$ such that for all non–zero $f \in J$, the maximum value of the dot product $\xi \cdot \alpha$ as α runs over $s(f)$ is assumed at least twice.

Bergman shows in [**3**] that the second and third descriptions are equivalent, and it follows from Bieri and Groves [**4**] that all descriptions are equivalent:

THEOREM 3.1 (Bergman, Bieri–Groves). $V_\infty^{(a)} = V_\infty^{(b)} = V_\infty^{(c)}$.

We therefore write V_∞ for the logarithmic limit set of the variety V. The ideal generated by a set of polynomials $\{f_i\}_i$ shall be denoted by $I(f_i)_i$, and the variety generated by the ideal I is denoted by $V(I)$ or $V(f_i)_i$, if $I = I(f_i)_i$.

3.1.2. Newton Polytopes. The third description of the logarithmic limit set can be illustrated using spherical duals of Newton polytopes. Let $f \in \mathbb{C}[X^{\pm}]$ be given as an expression $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha X^\alpha$. Then the *Newton polytope* of f is the convex hull of $s(f) = \{\alpha \in \mathbb{Z}^m \mid a_\alpha \neq 0\}$. Thus, if $s(f) = \{\alpha_1, \ldots, \alpha_k\}$, then

$$Newt(f) = Conv(s(f)) = \left\{ \sum_{i=1}^{k} \lambda_i \alpha_i \mid \lambda \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{3.3}$$

Note that $Newt(0) = \emptyset$, and since $V(0) = (\mathbb{C} - \{0\})^m$, we have $V(0)_\infty = S^{m-1}$.

For arbitrary subsets A, B of $\mathbb{R}^m$, let $A + B = \{a + b \mid a \in A, b \in B\}$. The following facts about Newton polytopes of polynomials are well known (cf. [**16**]):

1. Let $f, g \in \mathbb{C}[X^{\pm}]$ such that $fg \neq 0$. Then $Newt(fg) = Newt(f) + Newt(g)$.
2. $Newt(f + g) \subseteq Conv(s(f) \cup s(g)) = Conv(Newt(f) \cup Newt(g))$.

3.1.3. Spherical Duals. The *spherical dual* of a bounded convex polytope P in $\mathbb{R}^m$ is the set of vectors ξ of length 1 such that the supremum $\sup_{\alpha \in P} \alpha \cdot \xi$ is achieved for more than one α, and it is denoted by $Sph(P)$. Geometrically, $Sph(f)$ consists of all outward unit normal vectors to the support planes of P which meet P in more than one point. If P is the Newton polytope of a non–zero polynomial f, then the spherical dual of $Newt(f)$ is also denoted by $Sph(f)$. The following lemma is immediate from the third description of the logarithmic limit set:

LEMMA 3.2. *Let V be an algebraic variety in $(\mathbb{C}-\{0\})^m$ defined by the ideal J. Then V_∞ is the intersection over all non–zero elements of J of the spherical duals of their Newton polytopes.*

Spherical duals are easy to visualise in low dimensions. Given a convex polygon P in $\mathbb{R}^2$, its spherical dual is the collection of points on the unit circle defined by outward pointing unit normal vectors to edges of P. Given a convex polyhedron in $\mathbb{R}^3$, we obtain the vertices of its spherical dual again as points on the unit sphere S^2 arising from outward pointing unit normal vectors to faces. We then join two of these points along the shorter geodesic arc in S^2 if the corresponding faces have a common edge. This gives a finite graph in S^2.

3.1.4. If the ideal J is principal, defined by an element $f \neq 0$, then Bergman states in [**3**] that the set $V(f)_\infty$ is precisely $Sph(f)$, but he does not prove it. Lash proves this in [**32**] for the special case where f is an irreducible Laurent polynomial in three complex variables using complex analysis and the first description of the logarithmic limit set. We give a proof of the general case using the following lemma.

LEMMA 3.3. *Let $f, g \in \mathbb{C}[X^{\pm}]$. If $fg \neq 0$, then $Sph(fg) = Sph(f) \cup Sph(g)$.*

PROOF. If $\xi \in Sph(f)$, then there are distinct $\alpha_0, \alpha_1 \in Newt(f)$ such that

$$\alpha_0 \cdot \xi = \alpha_1 \cdot \xi = \sup_{\alpha \in Newt(f)} \alpha \cdot \xi.$$

Since $Newt(g)$ is convex and bounded, there is $\beta_0 \in Newt(g)$ such that

$$\beta_0 \cdot \xi = \sup_{\beta \in Newt(g)} \beta \cdot \xi.$$

Since $Newt(fg) = Newt(f) + Newt(g)$, it follows that $\beta_0 + \alpha_0, \beta_0 + \alpha_1 \in Newt(fg)$ and

$$(\alpha_0 + \beta_0) \cdot \xi = (\alpha_1 + \beta_0) \cdot \xi = \sup_{\gamma \in Newt(fg)} \gamma \cdot \xi.$$

Thus, $Sph(f) \subseteq Sph(fg)$, and by symmetry $Sph(g) \subseteq Sph(fg)$.

Now then assume that $\xi \in Sph(fg)$. Then there are distinct γ_0 and γ_1 in $Newt(fg)$ such that

$$\gamma_0 \cdot \xi = \gamma_1 \cdot \xi = \sup_{\gamma \in Newt(fg)} \gamma \cdot \xi.$$

Furthermore, there are $\alpha_i \in Newt(f)$ and $\beta_i \in Newt(g)$ such that $\gamma_i = \alpha_i + \beta_i$. If $\alpha_0 \cdot \xi < \alpha_1 \cdot \xi$, then $\gamma_0 \cdot \xi < (\alpha_1 + \beta_0) \cdot \xi$ and $\alpha_1 + \beta_0 \in Newt(fg)$, contradicting the choice of γ_0. Thus, $\alpha_0 \cdot \xi = \alpha_1 \cdot \xi$, and similarly $\beta_0 \cdot \xi = \beta_1 \cdot \xi$. A similar argument shows:

$$\alpha_0 \cdot \xi = \alpha_1 \cdot \xi = \sup_{\alpha \in Newt(f)} \alpha \cdot \xi,$$

and similarly for the β_i. Since $\gamma_0 \neq \gamma_1$, either $\alpha_0 \neq \alpha_1$ or $\beta_0 \neq \beta_1$, and hence either $\xi \in Sph(f)$ or $\xi \in Sph(g)$. ■

PROPOSITION 3.4. *Let* $f \in \mathbb{C}[X^{\pm}]$. *If* $f \neq 0$, *then* $V(f)_\infty = Sph(f)$.

PROOF. Since each non–zero element of $I(f)$ is of the form fg for some non–zero $g \in \mathbb{C}[X^{\pm}]$, Lemma 3.2 and Lemma 3.3 yield:

$$V(f)_\infty = \bigcap_{0 \neq g \in \mathbb{C}[X^{\pm}]} Sph(fg) = \bigcap_{0 \neq g \in \mathbb{C}[X^{\pm}]} (Sph(f) \cup Sph(g)) = Sph(f),$$

since $Sph(1) = \emptyset$. ■

Logarithmic limit sets of varieties defined by a principal ideal are therefore completely determined by a generator of the ideal, and readily computable. If the ideal J is not principal, then $V(J)_\infty$ is the intersection of the spherical duals of the Newton polytopes of *finitely* many elements in J. This follows from Bieri and Groves [**4**]. However, there is presently no known algorithm determining a suitable finite set of elements of J. Conjecturally, a Gröbner basis may suffice.

3.1.5. As Bergman notes, the spherical dual of the convex hull of a finite subset $A \subset \mathbb{Z}^m$ of cardinality r is a finite union of convex spherical polytopes. It is the union over all $\alpha_0, \alpha_1 \in A$ of the set of ξ satisfying the $2r$ inequalities

$$\alpha_0 \cdot \xi \geq \alpha \cdot \xi \qquad \text{and} \qquad \alpha_1 \cdot \xi \geq \alpha \cdot \xi,$$

where α ranges over A. This may be useful for calculations, but it also shows that corners of the convex spherical polytopes which we consider have rational coordinate ratios, and points with rational coordinate ratios are dense in the polytope. Such a polytope is called a *rational convex spherical polytope.* We summarise further results by Bergman and Bieri–Groves as follows:

THEOREM 3.5 (Bergman, Bieri–Groves). *The logarithmic limit set V_∞ of an algebraic variety V in $(\mathbb{C} - \{0\})^m$ is a finite union of rational convex spherical polytopes. The maximal dimension for a polytope in this union is achieved by at least one member P in this union, and we have*

$$\dim_{\mathbb{R}} V_\infty = \dim_{\mathbb{R}} P = (\dim_{\mathbb{C}} V) - 1.$$

Bergman asks in [**3**] whether every polytope in the above union always has the same dimension. In Chapter 5 it is shown that the component in the eigenvalue variety of the Whitehead link complement which arises from the Dehn surgery component has logarithmic limit set consisting of one connected component composed of 1–dimensional polytopes and two zero dimensional components.

In order to illustrate the above theorem, a generic picture for the projection of a variety in $(\mathbb{C} - \{0\})^2$ into B^2 and its logarithmic limit set is given in Figure 3.1.

3.1.6. Curve finding lemma. We conclude this section with a lemma, which is useful for applications of Culler–Shalen theory.

LEMMA 3.6. *Let V be an algebraic variety in $(\mathbb{C} - \{0\})^m$, and $\xi \in V_\infty$ be a point with rational coordinate ratios. Then there is a curve C, i.e. an irreducible subvariety of complex dimension one, in V such that $\xi \in C_\infty$.*

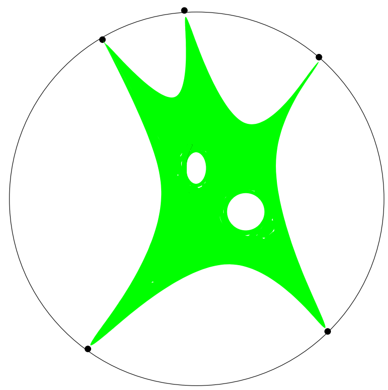


FIGURE 3.1. A logarithmic limit set in S^1

PROOF. By a lemma of Bergman ([**3**], p.464), the point ξ is contained in the logarithmic limit set of an irreducible component of V. Thus, we may assume that V is irreducible for the purpose of this proof.

Assume that V is defined by the ideal I in $\mathbb{C}[X^{\pm}] = \mathbb{C}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$. We follow word by word Bergman's construction starting after "(4)" on page 465, up to the sentence "Hence the points $u_y \log|y| \in B^m$ will approach $(0, \dots, 0, -1)$." on page 466. Here u_y denotes $(\sqrt{1 + \sum(\log(|y_i|)^2)})^{-1}$. Our adaption reads as follows:

> Let ξ be a point of V_∞ the ratios of whose coordinate functions are rational. Making use of the action of $GL_m(\mathbb{Z})$ on $\mathbb{C}[X^{\pm}]$ and the induced action on S^{m-1} (not isometric!), we can reduce to the case where $\xi = (0, \dots, 0, -1)$.
>
> Let R' designate the ring $\mathbb{C}[X_1^{\pm 1}, \dots, X_{m-1}^{\pm 1}; X_m] \subset R = \mathbb{C}[X^{\pm}]$, and R_0 the ring $\mathbb{C}[X_1^{\pm 1}, \dots, X_{m-1}^{\pm 1}]$, into which we map R' by sending X_m to 0. Let I' designate $I \cap R'$, and I_0 the image of I' in R_0. Thus, I_0 consists of the components of degree 0 in X_m of elements of I'. Let V' designate the subvariety of $(\mathbb{C} - \{0\})^{m-1} \times \mathbb{C}$ defined by I', and $V_0 = V' \cap (\mathbb{C} - \{0\})^{m-1} \times \{0\}$, the subvariety of $(\mathbb{C} - \{0\})^{m-1} \times \{0\}$ defined by I_0. V' will be the Zariski closure of V in $(\mathbb{C} - \{0\})^{m-1} \times \mathbb{C}$, and $V = V' - V_0$.
>
> If $(0, \dots, 0, -1) \in V_\infty$, this means that every nonzero $a \in I$ has more than one term of minimal degree in X_m. Hence every nonzero element of

> I_0 has more than one term, i.e. I_0 is a proper ideal of R_0. Then V_0 is non–empty: let $x \in V_0$.
>
> Now V will in fact be dense in V' under the topology induced by the absolute value on $\mathbb{C}$. [...] Hence we can obtain x as a limit of points $y \in V$. As y approaches x, $\log|y_i|$ will approach the finite values $\log|x_i|$ for $i = 1, \dots, m-1$, but $\log|y_m|$ will approach $-\infty$. Hence the points $u_y \log|y| \in B^m$ will approach $(0, \dots, 0, -1)$.

We construct a curve C in V such that $\xi = (0, \dots, 0, -1) \in C_\infty$ using the following two facts from [**34**]:

1. (1.14) Let Y be an irreducible affine variety. If X is a proper subvariety of Y, then $\dim X < \dim Y$.
2. (3.14) If X is an r–dimensional affine variety and $f_1, \dots, f_k$ are polynomial functions on X, then every component of $X \cap V(f_1, \dots, f_k)$ has dimension $\geq r - k$.

Since V_0 is a proper subvariety of V', obtained by intersection with a hyperplane, we have $\dim V_0 = \dim V' - 1$, and since $V = V' - V_0$, we have $\dim V = \dim V'$. We need to find a curve C' in V' which has non–trivial intersection with both V_0 and V, since then $C = C' \cap V$ is a curve in V such that $\xi \in C_\infty$.

Let $\dim V = d$. Let $x \in V_0$ and choose a polynomial f_1 on V_0 such that $f_1(x) = 0$, but f_1 is not identical to 0 on V_0. Then f_1 is not identical to 0 on V' and each component of the variety $V' \cap V(f_1)$ has dimension exactly $d-1$, and intersects V_0 in a $(d-2)$–dimensional subvariety. Let V_1 be a component of $V' \cap V(f_1)$ containing x.

Choose a polynomial f_2 such that $f_2(x) = 0$, but f_2 is not identical to 0 on $V_1 \cap V_0$. Then each component of $V_1 \cap V(f_2) = V' \cap V(f_1, f_2)$ has dimension exactly $d-2$ and intersects V_0 in a $(d-3)$–dimensional subvariety. We let V_2 be a component of $V' \cap V(f_1, f_2)$ containing x.

We now proceed inductively for $k = 2, \dots, d-2$. Given a component V_k of $V' \cap V(f_1, \dots, f_k)$ which is a $(d-k)$–dimensional subvariety of V', with the property that it contains x, but also points of $V' - V_0$, we let f_{k+1} be a polynomial function on V_k which vanishes on x but not on all of $V_k \cap V_0$. Then each component of $V' \cap V(f_1, \ldots, f_{k+1})$ has dimension exactly $d-k-1$ and intersects V_0 in a $(d-k-2)$–dimensional subvariety, and we let V_{k+1} be a component of $V' \cap V(f_1, \ldots, f_{k+1})$ containing x.

Then $V_{d-1} \cap V$ is a curve in V with the desired property. ■

3.2. The eigenvalue variety

Given a compact, orientable, irreducible 3–manifold with boundary consisting of a single torus, one calls boundary slopes of essential surfaces associated to ideal points of the character variety *strongly detected*. It is shown in [**11**] that the slope of each side of the Newton polygon of the A–polynomial is a strongly detected boundary slope.

We now generalise the A–polynomial to define an eigenvalue variety of a manifold with boundary consisting of h tori. We describe the *strongly detected boundary curves*, i.e. the boundary curves of essential surfaces associated to ideal points of the character variety, via a map from the logarithmic limit set of the eigenvalue variety to the $(2h-1)$–sphere.

3.2.1. We recall the following definitions and facts from algebraic geometry. Let V be a variety defined by an ideal I of $\mathbb{C}[X] = \mathbb{C}[X_1, \ldots, X_m]$. Then $\mathbb{C}[V] = \mathbb{C}[X]/I$. A map $\Phi : V \to W$ between affine varieties is called *dominating* if it is a polynomial map such that the smallest subvariety of W containing $\Phi(V)$ is W itself. If $\Phi : V \to W$ is a dominating map, then the induced ring homomorphism $\Phi_* : \mathbb{C}[W] \to \mathbb{C}[V]$ is injective (see [**16**], p.472).

Elements $f_1, \ldots, f_r \in \mathbb{C}[V]$ are called *algebraically independent over* $\mathbb{C}$ if there is no polynomial $P \neq 0$ of r variables with coefficients in $\mathbb{C}$ such that $P(f_1, \ldots, f_r) = 0$ in $\mathbb{C}[V]$. The dimension of V equals the maximal number of elements in $\mathbb{C}[V]$ which are algebraically independent over $\mathbb{C}$ (see [**16**], p.466).

3.2.2. ∂–incompressible. The following definition and facts can be found in [**26**]. A (properly embedded) surface S in a compact 3–manifold M is *∂–incompressible* if for each disc $D \subset M$ such that ∂D splits into two arcs α and β meeting only at their common endpoints with $D \cap S = \alpha$ and $D \cap \partial M = \beta$ there is a disc $D' \subset S$ with $\alpha \subset \partial D'$ and $\partial D' - \alpha \subset \partial S$.

A surface S is incompressible and ∂–incompressible if and only if each component of S is incompressible and ∂–incompressible.

LEMMA 3.7. [**26**] *Let S be a connected incompressible surface in the irreducible 3–manifold M, with ∂S contained in torus boundary components of M. Then either S is ∂–incompressible or S is a boundary parallel annulus.*

For the remainder of this section, we let M be a compact, orientable, irreducible 3–manifold with non–empty boundary consisting of a disjoint union of h tori. It follows from Definition 1.11 and Lemma 3.7 that an essential surface in M is ∂–incompressible.

3.2.3. Boundary curve space. Denote the boundary tori of M by $T_1, \ldots, T_h$, and choose an oriented meridian $\mathcal{M}_i$ and an oriented longitude $\mathcal{L}_i$ for each T_i. Let S be an incompressible and ∂-incompressible surface with non–empty boundary in M. There are coprime integers p_i, q_i such that S meets T_i in n_i curves parallel to $p_i\mathcal{M}_i + q_i\mathcal{L}_i$. We thus obtain a point

$$(n_1p_1, n_1q_1, \ldots, n_hp_h, n_hq_h) \in \mathbb{Z}^{2h} - \{0\}.$$

We view the above point as an element in $\mathbb{R}P^{2h-1}$ since most methods of detecting essential surfaces only do so up to projectivisation. Furthermore, we ignore the orientation of ∂S, so the loop $p_i\mathcal{M}_i + q_i\mathcal{L}_i$ is equivalent to $-p_i\mathcal{M}_i - q_i\mathcal{L}_i$. All we have to know are the relative signs of p_i and q_i. This equivalence induces a $\mathbb{Z}_2^{h-1}$ action on $\mathbb{R}P^{2h-1}$, and the closure of the set of *projectivised boundary curve* *coordinates*:

$$[n_1p_1, n_1q_1, \ldots, n_hp_h, n_hq_h] \in \mathbb{R}P^{2h-1}/\mathbb{Z}_2^{h-1}$$

arising from incompressible and ∂–incompressible surfaces with non–empty boundary in M is called the *boundary curve space* of M and denoted by $\mathfrak{BC}(M)$. Note that $\mathbb{R}P^{2h-1}/\mathbb{Z}_2^{h-1} \cong S^{2h-1}$ (see [**25**]).

THEOREM 3.8. [**25**] *Let M be a compact, orientable, irreducible 3–manifold with non–empty boundary consisting of a disjoint union of h tori.*

Then $\dim_{\mathbb{R}} \mathfrak{BC}(M) < h$.

3.2.4. Eigenvalue variety. Given a presentation of $\pi_1(M)$ with a finite number, n, of generators, $\gamma_1, \ldots, \gamma_n$, we introduce four affine coordinates (representing matrix entries) for each generator, which we denote by g_{ij} for $i = 1, \ldots, n$ and $j = 1, 2, 3, 4$. We view $\mathfrak{R}(M)$ as a subvariety of $\mathbb{C}^{4n}$ defined by an ideal J in $\mathbb{C}[g_{11}, \ldots, g_{n4}]$. Recall that there are elements $I_\gamma \in \mathbb{C}[g_{11}, \ldots, g_{n4}]$ for each $\gamma \in \pi_1(M)$ such that $I_\gamma(\rho) = \operatorname{tr}\rho(\gamma)$ for each $\rho \in \mathfrak{R}(M)$.

We identify $\mathcal{M}_i$ and $\mathcal{L}_i$ with generators of $\operatorname{im}(\pi_1(T_i) \to \pi_1(M))$. Thus, $\mathcal{M}_i$ and $\mathcal{L}_i$ are words in the generators for $\pi_1(M)$, and we define the following polynomial equations in the ring $\mathbb{C}[g_{11}, \ldots, g_{n4}, m_1^{\pm1}, l_1^{\pm1}, \ldots, m_h^{\pm1}, l_h^{\pm1}]$:

$$I_{\mathcal{M}_i} = m_i + m_i^{-1}, \tag{3.4}$$

$$I_{\mathcal{L}_i} = l_i + l_i^{-1}, \tag{3.5}$$

$$I_{\mathcal{M}_i\mathcal{L}_i} = m_il_i + m_i^{-1}l_i^{-1}, \tag{3.6}$$

for $i = 1, \ldots, h$. Let $\mathfrak{R}_E(M)$ be the variety in $\mathbb{C}^{4n} \times (\mathbb{C} - \{0\})^{2h}$ defined by J together with the above equations. For each $\rho \in \mathfrak{R}(M)$, equations (3.4– 3.6) have a solution since commuting elements of $SL_2(\mathbb{C})$ always have a common invariant subspace. The natural projection $p_1 : \mathfrak{R}_E(M) \to \mathfrak{R}(M)$ is therefore onto, and p_1 is a dominating map.

If $(a_1, \ldots, a_{4n}) \in \mathfrak{R}(M)$, then there is a $\mathbb{Z}_2^h$ action on the resulting points $(a_1, \ldots, a_{4n}, m_1, l_1, \ldots, m_h, l_h) \in \mathfrak{R}_E(M)$ by inverting both entries of a tuple (m_i, l_i) to (m_i^{-1}, l_i^{-1}). The group $\mathbb{Z}_2^h$ acts transitively on the fibres of p_1. The map p_1 is therefore finite–to–one with degree $\leq 2^h$. The maximal degree is in particular achieved when the interior of M admits a complete hyperbolic structure of finite volume, since Theorem 1.3 implies that there are points in $\mathfrak{X}_0(M) - \cup_{i=1}^h \{I^2_{\mathcal{M}_i} = 4\}$.

The *eigenvalue variety* $\mathfrak{E}(M)$ is the closure of the image of $\mathfrak{R}_E(M)$ under projection onto the coordinates $(m_1, \ldots, l_h)$. It is therefore defined by an ideal of the ring $\mathbb{C}[m_1^{\pm 1}, l_1^{\pm 1}, \ldots, m_h^{\pm 1}, l_h^{\pm 1}]$ in $(\mathbb{C} - \{0\})^{2h}$.

Note that this construction factors through a variety $\mathfrak{X}_E(M)$, which is the character variety with its coordinate ring appropriately extended, since it is shown in [**23**], that we can choose coordinates of $\mathfrak{X}(M)$ such that I_γ (as a function on $\mathfrak{X}(M)$) is an element of the coordinate ring of $\mathfrak{X}(M)$ for each $\gamma \in \pi_1(M)$ (see Section 1.1.2). We let $\mathrm{t}_E : \mathfrak{R}_E(M) \to \mathfrak{X}_E(M)$ be the natural quotient map, which is equal to $\mathrm{t} : \mathfrak{R}(M) \to \mathfrak{X}(M)$ on the first $4n$ coordinates, and equal to the identity on the remaining $2h$ coordinates.

There also is a restriction map $r : \mathfrak{X}(M) \to \mathfrak{X}(T_1) \times \ldots \times \mathfrak{X}(T_h)$, which arises from the inclusion homomorphisms $\pi_1(T_i) \to \pi_1(M)$, and we therefore denote the map $\mathfrak{X}_E(M) \to \mathfrak{E}(M)$ by r_E. Denote the closure of the image of r by $\mathfrak{X}_\partial(M)$. There is the following commuting diagram of dominating maps:

$$\begin{array}{ccccc} \mathfrak{R}_E(M) & \xrightarrow{\mathrm{t}_E} & \mathfrak{X}_E(M) & \xrightarrow{r_E} & \mathfrak{E}(M) \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle p_3}\downarrow \\ \mathfrak{R}(M) & \xrightarrow{\mathrm{t}} & \mathfrak{X}(M) & \xrightarrow{r} & \mathfrak{X}_\partial(M) \end{array}$$

Note that the maps p_1, p_2, t and t_E have the property that every point in the closure of the image has a preimage, and that the maps p_1, p_2 and p_3 are all finite–to–one of the same degree.

Recall the construction by Culler and Shalen. We start with a curve $C \subset \mathfrak{X}(M)$ and an irreducible component R_C in $\mathfrak{R}(M)$ such that $\mathrm{t}(R_C) = C$ to obtain the tautological representation $\mathcal{P} : \pi_1(M) \to SL_2\mathbb{C}(R_C)$. Let R'_C be an irreducible component of $\mathfrak{R}_E(M)$ with the property that $p_1(R'_C) = R_C$. Then $\mathbb{C}(R'_C)$ is a finitely generated extension of $\mathbb{C}(R_C)$, and we may think of $\mathcal{P}$ as a representation $\mathcal{P} : \pi_1(M) \to SL_2\mathbb{C}(R'_C)$.

If $\overline{r_E\, \mathrm{t}_E(R'_C)}$ contains a curve E, then $\mathbb{C}(R'_C)$ is a finitely generated extension of $\mathbb{C}(E)$, and hence to each ideal point of E we can associate essential surfaces using $\mathcal{P}$. Since the eigenvalue of at least one peripheral element blows up at an ideal point of E, the associated surfaces necessarily have non–empty boundary (see Section 1.2.11).

Thus, if there is a closed essential surface associated to an ideal point of C, then either $\overline{r_E\, \mathrm{t}_E(R'_C)}$ is 0–dimensional, or there is a neighbourhood U of an ideal point ξ of R'_C such that there are (finite) points in $\overline{r_E\, \mathrm{t}_E(U)} - r_E\, \mathrm{t}_E(U)$. The later are called *holes in the eigenvalue variety.*

3.2.5. Boundary slopes lemma. We denote the logarithmic limit set of $\mathfrak{E}(M)$ by $\mathfrak{E}_\infty(M)$. There are symmetries of the eigenvalue variety which give rise to symmetries of its logarithmic limit set. If $(m_1, l_1, \ldots, m_i, l_i, \ldots, m_h, l_h)$ is a point on $\mathfrak{E}(M)$, so is $(m_1, l_1, \ldots, m_i^{-1}, l_i^{-1}, \ldots, m_h, l_h)$ for any i. It follows that if $\xi = (x_1, \ldots, x_{2i-1}, x_{2i}, \ldots, x_{2h}) \in \mathfrak{E}_\infty(M)$, then $(x_1, \ldots, -x_{2i-1}, -x_{2i}, \ldots, x_{2h}) \in \mathfrak{E}_\infty(M)$ for $i = 1, \ldots, h$. If we factor by these symmetries, we obtain a quotient of the logarithmic limit set in $\mathbb{R}P^{2h-1}/\mathbb{Z}_2^{h-1} \cong S^{2h-1}$. The quotient map extends to a map $\Psi : S^{2h-1} \to S^{2h-1}$ of spheres, which has degree 2^h. Let

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and let T_h be the block diagonal matrix with h copies of T on its diagonal. Then T_h is orthogonal, and its restriction to S^{2h-1} is a map of degree one.

LEMMA 3.9. *Let M be an orientable, irreducible, compact 3–manifold with non–empty boundary consisting of a disjoint union of h tori. If $\xi \in \mathfrak{E}_\infty(M)$ is a point with rational coordinate ratios, then there is an essential surface with boundary in M whose projectivised boundary curve coordinate is $\Psi(T_h\xi)$.*

Note that the composite ΨT_h is a smooth map of degree 2^h. Since the set of rational points is dense in the logarithmic limit set, the image $\Psi(T_h \mathfrak{E}_\infty(M))$ is a closed subset of $\mathfrak{BC}(M)$.

3.2.6. Remark. For a manifold M with boundary consisting of a single torus, Lemma 3.9 coincides with the result of [**11**] that slopes of sides of the Newton polygon of the A–polynomial are strongly detected boundary slopes of M.

In this case, we may normalise $\mathfrak{BC}(M)$ and $\mathfrak{E}_\infty(M)$ such that their elements are pairs of coprime integers. Assume that there is a side of the Newton polygon of slope $-p/q$ where $p, q \geq 0$ with outward normal vector $\xi = (q,p)^T$. Then $\xi \in \mathfrak{E}_\infty(M)$ according to Section 3.1.3. We now perform matrix multiplication:

$$T\xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix},$$

and arrive at a point $[p, -q] \in \mathfrak{BC}(M)$. The analysis for the remaining cases is similar, and we have recovered the known relationship.

3.2.7. Proof of Lemma 3.9. The coordinate ratios of ξ are rational, and hence Lemma 3.6 provides a curve C in $\mathfrak{E}(M)$ such that $\xi \in C_\infty$. From description (3.2) of elements in the logarithmic limit set, we know that ξ encodes the definition of a real–valued valuation on the function field of C:

$$\xi = (-v(m_1), \ldots, -v(l_h)).$$

Since the coordinate ratios of ξ are rational, we may rescale the valuation by $r > 0$, such that for all i, $rv(m_i)$ and $rv(l_i)$ are integers, and such that there is $\alpha \in \mathbb{Z}^{2h} - \{0\}$ with $rv(m_1^{\alpha_1} \cdots l_h^{\alpha_{2h}}) = 1$. Thus, rv is a normalised, discrete, rank 1 valuation of $\mathbb{C}(C)$. We finish the proof with standard arguments. For details see Section 2.1 of [**32**], the proof of Proposition 2.3 of [**20**] or Section 5.6 of [**11**].

The construction by Culler and Shalen yields an action of $\pi_1(M)$ on a simplicial tree $\mathcal{T}$ associated to a finitely generated extension F of $\mathbb{C}(C)$ and to a valuation $w = dv$ for some $d > 0$. The translation length of $\gamma \in \pi_1(M)$ with eigenvalue e (as an element of F) is $\ell(\gamma) = |2w(e)|$ (see [**42**], Section 3.9). A dual essential surface S associated to the action can be chosen such that the geometric intersection number of a loop in M representing γ with S is equal to $\ell(\gamma)$ for all $\gamma \in \mathrm{im}(\pi_1(T_i) \to \pi_1(M))$. This implies that a boundary curve of S on T_i is homotopic to one of $p_i\mathcal{M}_i \pm q_i\mathcal{L}_i$, where $n_ip_i = |2w(l_i)|$ and $n_iq_i = |2w(m_i)|$ and p_i, q_i are coprime. We need to determine the relative sign.

If $p_i\mathcal{M}_i + q_i\mathcal{L}_i$ represents an element of the subgroup $\mathrm{im}(\pi_1(\partial S) \to \pi_1(M))$, it is contained in the stabilizer of an edge, and hence has translation length equal to 0 since the action is without inversions. The elements $\mathcal{M}_i$ and $\mathcal{M}_i^{-1}$ translate points in opposite directions along their axis. Since $\mathcal{L}_i$ commutes with $\mathcal{M}_i$, they have a common axis. Thus, the curve $p_i\mathcal{M}_i + q_i\mathcal{L}_i$ can have zero translation length only if $\mathcal{M}_i$ and $\mathcal{L}_i$ translate in opposite directions. This is the case if and only if $w(m_i)$ and $w(l_i)$ have opposite signs. Thus, the relative sign of p_i and q_i is opposite the relative sign of $w(m_i)$ and $w(l_i)$. We conclude that a boundary curve of S on T_i corresponds to the unoriented homotopy class of the element $\frac{1}{n_i}(2w(l_i)\mathcal{M}_i - 2w(m_i)\mathcal{L}_i)$. Thus, the boundary curves of S are encoded by:

$$\begin{aligned} &(2w(l_1), -2w(m_1), \ldots, 2w(l_h), -2w(m_h)) \\ =&2d(v(l_1), -v(m_1), \ldots, v(l_h), -v(m_h)) \\ =&2d(T_h\xi) \end{aligned}$$

This completes the proof. ■

3.2.8. Dimension of the eigenvalue and character varieties. We now deduce some bounds on the dimension of the eigenvalue and the character variety.

PROPOSITION 3.10. *Let M be an orientable, irreducible, compact 3–manifold with boundary consisting of h tori. Then* $\dim_{\mathbb{C}} \mathfrak{E}(M) \leq h$.

Moreover, if the interior of M admits a complete hyperbolic structure of finite volume, then $\dim_{\mathbb{C}} \mathfrak{E}(M) = h$.

PROOF. We have $h > \dim_{\mathbb{R}} \mathfrak{BC}(M) \geq \dim_{\mathbb{R}} \mathfrak{E}_\infty = \dim_{\mathbb{C}} \mathfrak{E}(M) - 1$, where the first inequality is due to Theorem 3.8, the second to Lemma 3.9, and the equality to Theorem 3.5. It follows that $h \geq \dim_{\mathbb{C}} \mathfrak{E}(M)$.

To prove the second statement, it is enough to show that under the assumptions, there is a subvariety of $\mathfrak{E}(M)$ of dimension $\geq h$. Theorem 1.3 asserts that there is a component $\mathfrak{X}_0(M)$ of $\mathfrak{X}(M)$ of complex dimension equal to h. Thurston shows in [**44**], Section 5.8, that the map $\mathfrak{X}_0(M) \to \mathbb{C}^h$ defined by $\chi_\rho \to (I_{\mathcal{M}_1}(\rho), \ldots, I_{\mathcal{M}_h}(\rho))$ maps a small open neighbourhood of χ_0 to an open neighbourhood of $(\pm 2, \ldots, \pm 2)$. This is only possible if $I_{\mathcal{M}_i} = \operatorname{tr} \rho(\mathcal{M}_i)$ are algebraically independent over $\mathbb{C}$ as elements of $\mathbb{C}[X_0]$. There is a component X_0' of $\mathfrak{X}_E(M)$ with the same properties, since p_2 is a finite dominating map and the induced homomorphism $p_{2*} : \mathbb{C}[X_0] \to \mathbb{C}[X_0']$ is injective.

Let $E_0 = \overline{r_E \, \mathrm{t}_E(X_0')}$. If the functions $m_i + m_i^{-1}$, $i = 1, \ldots, h$, are algebraically dependent over $\mathbb{C}$ as elements of $\mathbb{C}[E_0]$, then they are also algebraically dependent over $\mathbb{C}$ as elements of $\mathbb{C}[X_0']$ since we have a ring homomorphism $\mathbb{C}[E_0] \to \mathbb{C}[X_0']$. But the identities (3.4) imply that then the functions $I_{\mathcal{M}_i}$ are algebraically dependent over $\mathbb{C}$ as elements of $\mathbb{C}[X_0']$, contradicting the fact that p_{2*} is injective.

Thus, the functions $m_i + m_i^{-1}$, $i = 1, \ldots, h$, are algebraically independent over $\mathbb{C}$ as elements of $\mathbb{C}[E_0]$, and hence $\dim_{\mathbb{C}} E_0 \geq h$ according to ([**16**], 9.5 Theorem 2). This completes the proof. ■

COROLLARY 3.11. *Let M be an orientable, irreducible, compact 3–manifold with boundary consisting of h tori. If M contains no closed essential surface, then* $\dim_{\mathbb{C}} \mathfrak{X}(M) = \dim_{\mathbb{C}} \mathfrak{E}(M) \leq h$.

PROOF. We have $\dim_{\mathbb{C}} \mathfrak{X}(M) = \dim_{\mathbb{C}} \mathfrak{X}_E(M)$. Assume there is a component X of $\mathfrak{X}_E(M)$ such that $\dim X > \dim \overline{r_E(X)}$. Then [**34**], Theorem (3.13), implies that there is a subvariety V of dimension ≥ 1 in X which maps to a single point of $\mathfrak{E}(M)$. Assocated to an ideal point of a curve in V, we find a closed essential surface since the traces of all peripheral elements are constant on V. This contradicts our assumption on M. Thus, for all components X of $\mathfrak{X}(M)$, we have $\dim X = \dim \overline{r_E(X)}$, and this proves the claim. ■

3.2.9. Remark. Let M be an orientable, irreducible, compact 3–manifold with boundary consisting of h tori, such that the interior of M admits a complete hyperbolic structure of finite volume. The proof of Proposition 3.10 implies that the restriction of r to the Dehn surgery component, $r_0 : X_0 \to \mathfrak{X}_\partial(M)$, is finite–to–one onto its image. The degree of r_0 depends upon $H^1(M, \mathbb{Z}_2)$, so let us consider the corresponding map $\overline{r}_0 : \overline{X}_0 \to \overline{\mathfrak{X}}_\partial(M)$. In the case where $h = 1$, Dunfield has shown in [**20**] that $\overline{r}_0$ has degree 1 onto its image using Thurston's Hyperbolic Dehn Surgery Theorem and a Volume Rigidity Theorem (Gromov–Thurston–Goldman). It would be interesting to generalise Dunfield's proof to arbitrary h, showing that $\overline{r}_0$ is always a birational isomorphism onto its image.

3.2.10. $\mathbf{PSL_2}(\mathbb{C})$–eigenvalue variety. Analogous to $\mathfrak{E}(M)$, one can define a $PSL_2(\mathbb{C})$–eigenvalue variety $\overline{\mathfrak{E}}(M)$, since the function $I^2_\gamma : \overline{\mathfrak{X}}(M) \to \mathbb{C}$ defined by $I^2_\gamma(\overline{\rho}) = (\operatorname{tr} \overline{\rho}(\gamma))^2$ is regular (i.e. polynomial) for all $\gamma \in \pi_1(M)$ (see [**6**]). Thus, we construct a variety $\overline{\mathfrak{R}}_E(M)$ using the relations

$$I^2_{\mathcal{M}_i} = M_i + 2 + M_i^{-1},$$
$$I^2_{\mathcal{L}_i} = L_i + 2 + L_i^{-1},$$
$$I^2_{\mathcal{M}_i\mathcal{L}_i} = M_iL_i + 2 + M_i^{-1}L_i^{-1},$$

for $i = 1, \ldots, h$.

Consider the representation of $\mathbb{Z} \oplus \mathbb{Z}$ into $PSL_2(\mathbb{C})$ generated by the images of

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.7}$$

In $PSL_2(\mathbb{C})$, the image of this representation is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, but the image of any lift to $SL_2(\mathbb{C})$ is isomorphic to the quaternion group Q_8 (in Quaternion group notation). If $\overline{\rho} \in \overline{\mathfrak{R}}(M)$ restricts to such an irreducible abelian representation on a boundary torus T_i, then the equations $M_i + 2 + M_i^{-1} = 0$, $L_i + 2 + L_i^{-1} = 0$, $M_i L_i + 2 + M_i^{-1} L_i^{-1} = 0$ have no solution. In particular, the projection $\overline{p} : \overline{\mathfrak{R}}_E(M) \to \overline{\mathfrak{R}}(M)$ may not be onto.

The closure of the image of $\overline{\mathfrak{R}}_E(M)$ onto the coordinates $(M_1, L_1, \ldots, M_h, L_h)$ is denoted by $\overline{\mathfrak{E}}(M)$ and called the $PSL_2(\mathbb{C})$*-eigenvalue variety.* The relationship between ideal points of $\overline{\mathfrak{E}}(M)$ and strongly detected boundary curves is the same as for the $SL_2(\mathbb{C})$–version, since the proof of Lemma 3.9 applies without change:

Lemma 3.12. *Let M be an orientable, irreducible, compact 3–manifold with non–empty boundary consisting of a disjoint union of h tori. If $\xi \in \overline{\mathfrak{E}}_\infty(M)$ is a point with rational coordinate ratios, then there is an essential surface with boundary in M whose projectivised boundary curve coordinate is $\Psi(T_h\xi)$.*

CHAPTER 4

Degenerations and normal surfaces

Throughout this chapter, M denotes the interior of an orientable, compact 3–manifold with non–empty boundary consisting of a disjoint union of tori, and $\mathcal{T}$ an ideal (topological) triangulation of M.

In Section 4.1, we describe normal surface Q–theory in M with respect to $\mathcal{T}$ following Kang [**31**]. In Section 4.2, we define a variety $\mathfrak{D}(M)$ which parametrises the hyperbolic structures on M obtained using $\mathcal{T}$, and we associate to each ideal point of $\mathfrak{D}(M)$ a projective class of normal surfaces. In Section 4.3, we use holonomies to show the existence of essential surfaces with certain boundary curves. In Section 4.4, we determine the dimension of $\mathfrak{D}(M)$, as well as the dimension of the projective solution space of normal surface Q–theory under the additional assumptions that M is a complete hyperbolic 3–manifold, and that $\mathcal{T}$ is an ideal hyperbolic triangulation of M. The reader may like to refer to the examples in Section 4.5 and Chapter 5 whilst reading Sections 4.1 to 4.4.

4.1. Normal surface Q–theory

In this section, we give an introduction to normal surface theory in 3–manifolds with ideal triangulations as it can be found in [**31**] and the documentation of [**50**]. We also decribe an algorithm to compute the boundary curves of normal surfaces.

4.1.1. Let M be the interior of an orientable, compact 3–manifold with non–empty boundary consisting of a disjoint union of tori. A 3–simplex with its vertices removed is called an *ideal tetrahedron*, and we refer to the removed vertices as *ideal vertices*. Similarly for its subsimplices. We will often drop the adjective "ideal". A (topological) *ideal triangulation* $\mathcal{T}$ of M is a decomposition of M into the union of ideal tetrahedra identified along faces, where we do not allow the identification of a face of a 3–simplex to itself. Such a triangulation always exists and is not unique (see [**5**]). Since the Euler characteristic of M is zero, the number of edges in $\mathcal{T}$ is equal to the number of ideal tetrahedra.

The ideal vertices of $\mathcal{T}$ are called the *ends* of M. Removing a small open neighbourhood of each end of M results in a compact submanifold M' of M whose boundary consists of tori. We call M' a *compact core* of M, and refer to its boundary tori also as the boundary tori of M. Each boundary torus T_i inherits a triangulation $\mathcal{T}_i$ from $\mathcal{T}$.

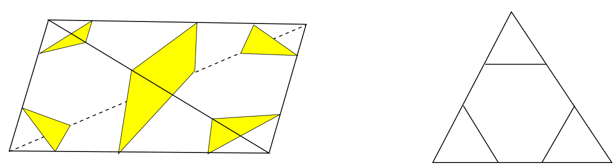


FIGURE 4.1. Normal discs in a tetrahedron and normal arcs in a triangle

An embedded surface S in M is said to be *normal* (with respect to $\mathcal{T}$) if it intersects each 3–simplex σ of $\mathcal{T}$ in a (possibly empty) disjoint union of discs, each of which is bounded by a quadrilateral (Q–disc) or a triangle (T–disc) of the type shown in Figure 4.1. These discs are called *elementary discs.* An isotopy of M is called *normal* (with respect to $\mathcal{T}$), if it leaves all simplices of $\mathcal{T}$ invariant.

Since $\mathcal{T}$ is an ideal triangulation, normal surfaces may intersect $\mathcal{T}$ in an infinite number of T–discs. Such surfaces are sometimes called *spun normal surfaces.* A normal surface in M is *properly embedded* if its intersection with any compact core is properly embedded in that compact core. The *boundary curves* of a normal surface S are the curves in the intersection of S with the boundary tori of a suitable compact core.

If Δ is a (possibly ideal) triangle, there are three *normal arc types* in Δ. An arc is called v–type if it separates the vertex v from the other two vertices of Δ. A path in T_i is called *normal* if it intersects each triangle in $\mathcal{T}_i$ in a (possibly empty) disjoint union of normal arcs.

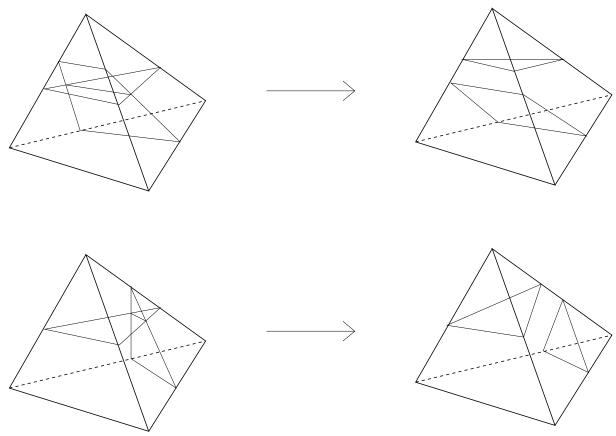
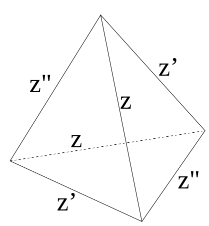
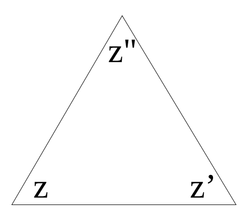


FIGURE 4.2. Regular exchanges

If two normal discs intersect transversely and not both are quadrilaterals, we may perform regular exchanges as shown in Figure 4.2 to obtain non–intersecting normal pieces. Note that a regular exchange between two normal discs results in normal discs of the same type.



(a) Ideal tetrahedron



(b) Link of a vertex

FIGURE 4.3. Convention for parameters

4.1.2. Let σ be an oriented ideal tetrahedron. We label the edges of σ with parameters z, z', z'', such that the labelling is as pictured in Figure 4.3. Thus, opposite edges have the same parameter, and the ordering z, z', z'' agrees with a right–handed screw orientation of σ. It follows that the labelling is uniquely determined once the parameter z is assigned to any edge of σ. The link $L(a)$ of an ideal vertex a of σ is an elementary disc Δ whose sides are a–type arcs on faces of σ and whose vertices inherit moduli from the edge parameters. We always view $L(a)$ from a.

There are three quadrilateral types in σ, and we denote them as follows. Let q denote the quadrilateral type which does not meet the edges labelled z, q' be the quadrilateral type disjoint from edges with label z', and q'' be the quadrilateral type disjoint from edges with label z''. The types are shown in Figure 4.4. We will often think of a quadrilateral type as a variable to which we assign an integer encoding the number of Q–discs of this type placed in σ.

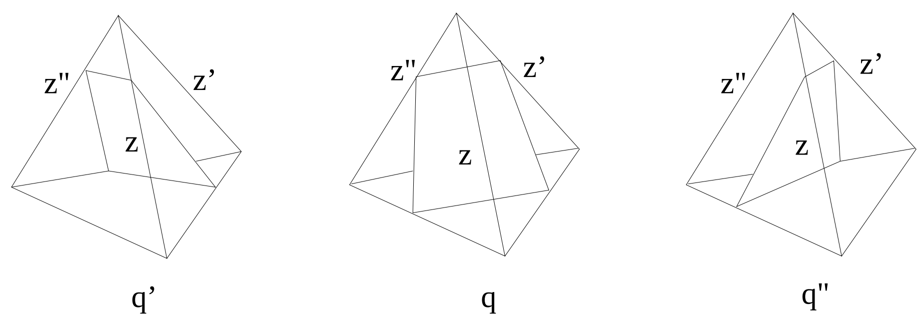


FIGURE 4.4. Convention for quadrilateral types

Let $\{\sigma_1, \ldots, \sigma_n\}$ be the ideal tetrahedra in the triangulation $\mathcal{T}$ of M. In an ideal triangulation, an edge of a tetrahedron occurs in different tetrahedra, and may also occur in the same tetrahedron with multiplicity. Denote by $\tilde{\sigma}_i$ an ideal 3–simplex (without self-identifications) which covers the simplex σ_i. Assign the parameter z_i to any edge of $\tilde{\sigma}_i$, and label the remaining edges in the unique resulting way with z_i, z_i', z_i'', and denote the quadrilateral types in $\tilde{\sigma}_i$ by q_i, q_i', q_i'' according to the above conventions. There is a 1–1 correspondence between quadrilateral types in $\tilde{\sigma}_i$ and in σ_i, and we will denote the quadrilateral types in σ_i by the same symbols. Since the quotient map $\tilde{\sigma}_i \to \sigma_i$ is a homeomorphism between the interiors of the

3–simplices, we will also use the labels z_i, z'_i, z''_i for "edges of σ_i" in the obvious way, and suppress any further reference to $\tilde{\sigma}_i$.

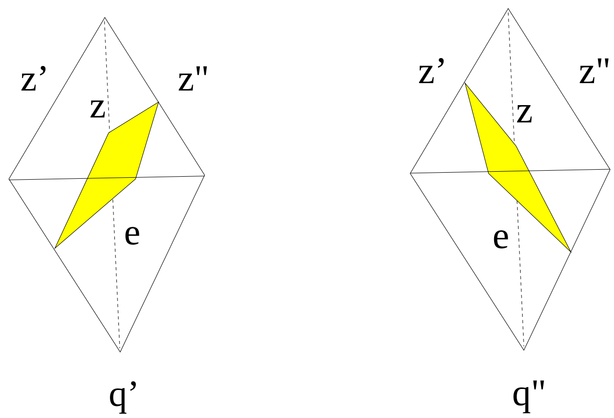
4.1.3. Normal Q–coordinates. A properly embedded normal surface S intersects $\mathcal{T}$ in normal discs, and hence inherts a cell structure. If $x_i^{(j)}$ is the number of Q–discs of type $q_i^{(j)}$ in this cell structure, then let $N(S) = (x_1, x'_1, x''_1, \ldots, x''_n)$ be the *normal Q–coordinate* of S. Thus, $N(S)$ is a point in $\mathbb{R}^{3n}$, and we label the coordinate axes in $\mathbb{R}^{3n}$ by the quadrilateral types.

Any properly embedded normal surface which consists entirely of triangles must be a collection of boundary parallel tori. We call these surfaces *boundary parallel.* The normal Q–coordinate of S satisfies the following two necessary conditions.

4.1.4. Admissible. A point $X = (x_1, x'_1, x''_1, \ldots, x''_n) \in \mathbb{R}^{3n}$ is called *admissible* if for each $i \in \{1, \ldots, n\}$ and each $j \in \{0, 1, 2\}$, $x_i^{(j)} \geq 0$ and at most one of x_i, x'_i, x''_i is non-zero. If S is embedded, its cell structure allows Q-discs of at most one quadrilateral type in each tetrahedron, and hence $N(S)$ is admissible.

4.1.5. Q–matching equations. To each edge in $\mathcal{T}$, we associate a so–called *Q–matching equation*, which is satisfied by the normal Q-coordinates of any properly embedded normal surface. This arises from the fact that a properly embedded surface intersects a small regular neighbourhood of the edge in a collection of discs, and that each of these discs is uniquely determined by its intersection with the boundary of the neighbourhood.

Consider the collection $\mathcal{C}$ of tetrahedra meeting at an edge e of $\mathcal{T}$, including k copies of σ if e occurs k times as an edge in σ. We form the *abstract neighbourhood* $B(e)$ of e by pairwise identifying faces of tetrahedra in $\mathcal{C}$ such that there is a well defined quotient map from $B(e)$ to the neighbourhood of e in M. $B(e)$ is a ball with finitely many points missing on its boundary. We think of the ideal endpoints a and b of e as the poles of its boundary sphere, and the remaining points as positioned on the equator, see Figure 4.6(a). Let σ be a tetrahedron in $\mathcal{C}$. The boundary square of a Q–disc $\square$ of type q in σ meets the equator of $\partial B(e)$ if and only if $\square$ has



(a) positive (b) negative

FIGURE 4.5. Slopes of quadrilaterals: q' has positive and q'' has negative slope on $\partial B(e)$.

a vertex on e. In this case, $\square$ has a slope of a well–defined sign on $\partial B(e)$ which is independent of the orientation of e. We use Figure 4.5 as a definition of *positive* and *negative slopes*.

Since opposite edges of σ have the same parameter, and since quadrilateral types are invariant under the Kleinian 4-group of rotational symmetries of σ, the slope of $\square$ on $\partial B(e)$ is uniquely determined by the parameter of e (with respect to σ). If S is properly embedded, then S intersects $B(e)$ in (possibly infinitely many) discs which are made up of elementary discs, and hence S intersects $\partial B(e)$ in a collection of circles. T–discs with corners on e do not meet the equator of $\partial B(e)$. Thus, the number of positive slopes of quadrilaterals in S on $\partial B(e)$ must be equal to the number of negative slopes, since otherwise the intersection of some Q–disc in S with $\partial B(e)$ is not contained in a circle, but in a helix on $\partial B(e)$.

It follows that the normal Q–coordinate of S satisfies a linear equation for each edge e in $\mathcal{T}$, which we call the *Q–matching equation of e*. The Q–matching equations can be described with respect to our parameterisation as follows.

From Figure 4.5 we deduce that if e_j is identified with an edge of σ_i which has parameter z_i, then the Q–discs of type q_i' have positive slope on $\partial B(e_j)$, and the Q–discs of type q_i'' have negative slope. Thus, if a_{ij} is the number of times e_j has label z_i, then the contribution to the Q–matching equation can be written as $a_{ij}(q_i' - q_i'')$. We define a_{ij}' and a_{ij}'' accordingly, and obtain the *Q–matching equation of e_j*:

$$0 = \sum_{i=1}^{n} (a_{ij}'' - a_{ij}')q_i + (a_{ij} - a_{ij}'')q_i' + (a_{ij}' - a_{ij})q_i''. \tag{4.1}$$

4.1.6. Projective solution space. Let $X = (x_1, \ldots, x_n'')$ be a non–zero solution to the Q–matching equations. Then αX is a solution for any $\alpha \in \mathbb{R}$. One therefore often considers the *projective solution space P(M)* to the Q–matching equations, which consists of all solutions X with the property that $1 = \sum_{i=1}^{n} x_i + x_i' + x_i''$ and $x_i^{(j)} \geq 0$ for all $i \in \{1, \ldots, n\}$ and $j \in \{0, 1, 2\}$. $P(M)$ is a convex polytope, and its vertices are called *vertex solutions*. The subset of admissible solutions in $P(M)$ is denoted by $\mathcal{N}(M)$. Note that for $N \in \mathcal{N}(M)$, there is a countable family of embedded normal surfaces S_i such that $N(S_i) = \alpha N$ for some $\alpha \in \mathbb{R}$. The set of all such surfaces is called a *projective class*, and we call a normal surface a *minimal representative*, if the corresponding scaling factor α is minimal.

THEOREM 4.1. [**31**] *Let M be an orientable non–compact 3–manifold with a topological ideal triangulation. For each integer solution $N \in \mathcal{N}(M)$ there exists a unique properly embedded normal surface S (possibly non–compact) in M with no boundary parallel components such that $N(S) = N$.*

We first sketch a proof of this theorem, as given by Jeff Weeks in the documentation of `SnapPea` [**50**] (see the file normal_surface_construction.c). In each tetrahedron σ_i of $\mathcal{T}$ place a finite number of parallel quadrilaterals of one chosen type, and four infinite stacks of parallel T–discs near the vertices. This determines a normal surface in σ_i which is unique up to isotopy.

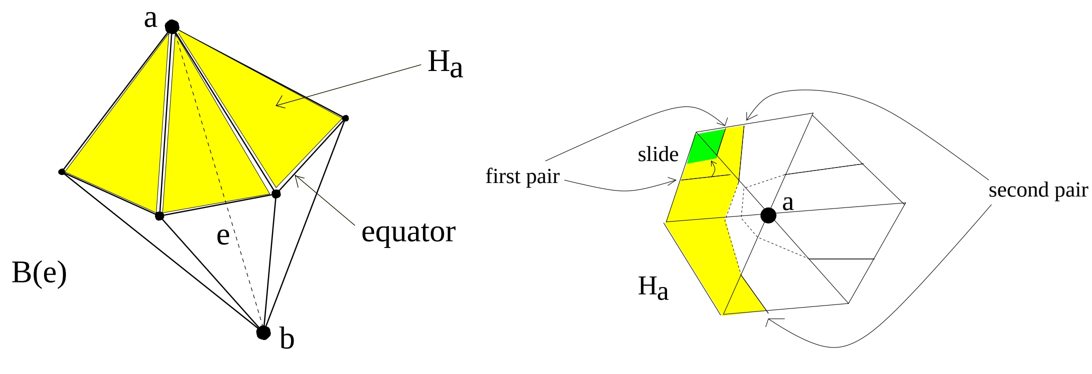
The intersection of this surface with each ideal triangle in the boundary of σ_i is topologically the same, and consists of three (infinite) families of normal arcs near the vertices. Given the combinatorial structure of $\mathcal{T}$, there is a unique way to identify faces of adjacent tetrahedra so the normal surfaces in the tetrahedra match up along the faces. This gives a normal surface in M minus the edges of $\mathcal{T}$.

We have seen in Subsection 4.1.5 that the Q–matching equations hold if and only if the result is a surface in the neighbourhood of each edge. Thus, each admissible solution to the Q–matching equations determines a normal surface S in a canonical way. Since a connected properly embedded surface which is entirely made up of triangles is a boundary parallel torus, discarding components of S which are entirely made up of triangles yields a normal surface without boundary parallel components.

4.1.7. We now describe a proof of Theorem 4.1 based on Ensil Kang's approach in [**31**]. Given a solution of the Q–matching equations, a normal surface is constructed as follows. First, a collection Ω of quadrilaterals as specified by the admissible solution N is chosen. A local analysis in the abstract neighbourhoods of edges of $\mathcal{T}$ determines how corners and edges of quadrilaterals are identified to obtain a 2–complex S'. This analysis also shows locally how triangles will be attached to unglued edges of quadrilaterals in S'.

In the second step, closed curves consisting of unglued edges of quadrilaterals are identified, and 2–complexes consisting entirely of triangles are attached to S' along these curves. It is then shown that one obtains a normal surface S after performing regular exchanges.

We follow [**31**] in the first step, and in the second step, we replace the "truncated projection maps" of [**31**] by a transverse orientation of curves on the induced triangulation of the boundary tori, to obtain additional information about the boundary components of the resulting surface.



(a) Abstract edge neighbourhood (b) Outermost pairs on H_a

FIGURE 4.6. Abstract neighbourhood and slopes on a hemisphere

4.1.8. Step 1: Q–corner gluing. Let N be an admissible solution to the Q–matching equations, and let Ω be a collection of Q–discs corresponding to N. Let e be an edge in $\mathcal{T}$ with ideal vertices a and b. Denote by H_a and H_b the hemispheres of $\partial B(e)$ containing a and b respectively, such that the link of e in $B(e)$ is the intersection of H_a and H_b, which we call the equator. (See Figure 4.6).

Each Q–disc which has positive or negative slope on $\partial B(e)$ contributes an arc to H_a, and we let $A = \{\alpha_i\}_i$ be the set of "positive" arcs and $B = \{\beta_i\}_i$ be the set of "negative" arcs. Then A and B have the same number of elements. We will define a bijection Φ_a between A and B, which corresponds to "identifying outermost arcs first, and then working successively inwards". This is described below and illustrated in Figure 4.6.

1. **Sliding pairs.** Assume that there are elements of A which have an endpoint on the same 1–simplex e' of H_a as elements of B. Assume $\alpha \in A$ and $\beta \in B$ have this property, and that we can slide the endpoints of α and β on e' to coincide without the arcs intersecting any other arcs except at endpoints. Then $\alpha \cup \beta$ together with a portion of the equator bounds a disc on H_a which does not contain a. If the interior of this disc does not contain any elements

of A and B, we call α and β an *outermost (sliding) pair*, and let $\Phi_a : \alpha \leftrightarrow \beta$. (See first pair in Figure 4.6).

Assume we have found an outermost sliding pair. Then repeat this step for the remaining arcs, replacing A and B by $A - \{\alpha\}$ and $B - \{\beta\}$ respectively, until the set of arcs in $A \cup B$ on which Φ_a is not defined does not contain positive and negative arcs with endpoints on a common edge.

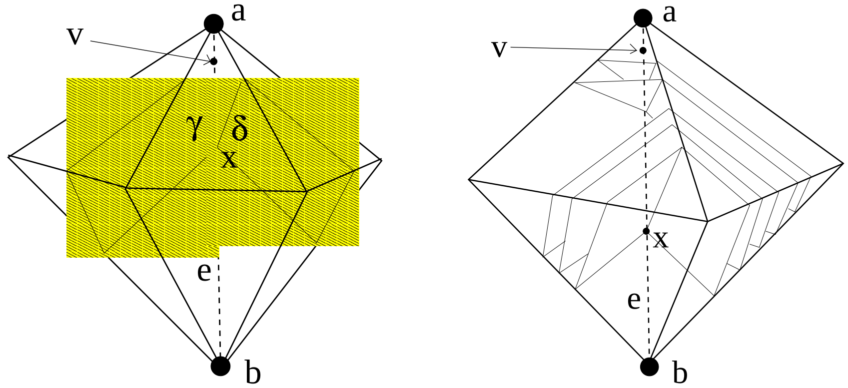
2. **Arc–adding pairs.** Let $A' \subseteq A$ and $B' \subseteq B$ be the subsets on which Φ_a is not defined after step 1. Assume we can join $\alpha \in A'$ and $\beta \in B'$ by a–type arcs in H_a such that the resulting path γ is normal and does not meet other elements of A' and B'. Consider the disc bounded by γ and a portion of the equator which does not contain a. If this disc does not contain any elements of $A' \cup B'$, we call α and β an *outermost (arc–adding) pair* and let $\Phi_a : \alpha \leftrightarrow \beta$. (See second pair in Figure 4.6).

   Assume we have found an outermost arc adding pair. Then repeat this step for the remaining arcs, until Φ_a is a 1-1 map between A and B.

Applying the procedure to both H_a and H_b gives us maps Φ_a and Φ_b. We identify the corners of two quadrilaterals $\square_1$ and $\square_2$ on e if they have sides α and β on $\partial B(e)$ such that either $\Phi_a : \alpha \leftrightarrow \beta$ or $\Phi_b : \alpha \leftrightarrow \beta$. This is the *$Q$–corner gluing rule* of [**49**].

Note that in the case of sliding pairs, we temporarily identify corners of quadrilaterals on a 1-simplex e' in H_a. In order to show that the Q–corner gluing rule is well–defined globally, one has to show that the same identification occurs in $B(e')$. This can be found in [**31**]. We conclude that edges of quadrilaterals are identified if and only if their endpoints are identified by the Q–corner gluing rule.

Denote by S' the 2–complex obtained by identifying corners and edges of quadrilaterals as forced by the corner gluing rule at each abstract edge neighbourhood. The a– and b–type arcs we added temporarily correspond to T–discs which will be attached to the quadrilaterals, see Figure 4.7. In the global analysis, we have to



(a) Example 1: $B(e)$. At x we have a pair of unglued a–type and a pair of unglued b–type arcs.

(b) Example 2: $B(e)$. At x, we only have a pair of unglued b–type arcs.

FIGURE 4.7. Q–corner gluing

take care of the fact that an edge of a quadrilateral in S' may occur several times in the the abstract neighbourhood of an edge, and it may occur in the abstract neighbourhoods of two different edges. Since each T–disc type is represented exactly once in the induced triangulation $\mathcal{T}_i$ of some boundary torus of M, the global analysis can be done using the induced triangulation of the boundary of M.

4.1.9. Step 2. Let e be an edge of $\mathcal{T}$ with endpoints at ideal vertices a and b. At a point x on e where corners of quadrilaterals are identified, we find unglued edges of quadrilaterals in pairs of a–type arcs and b–type arcs, see for example Figure 4.7. The second step in the proof of Theorem 4.1 consists of identifying closed curves in $\partial S'$ along which one glues 2–complexes consisting entirely of triangles to S'. We call them *T–surfaces*, and distinguish between two types.

A *finite* T–surface is normal isotopic to a subcomplex of $\mathcal{T}_i$ whose interior is homeomorphic to the interior of a disc or a once–punctured torus.

An *infinite* T-surface is homeomorphic to a half–open annulus, and can be constructed as follows. Let c be an essential simplicial curve on T_i and give it a transverse orientation. Take an infinite family of disjoint normal tori in the regular neighbourhood of the end corresponding to T_i. Denote the tori in this family by T_i^j, where $j = 1, 2, \dots$ There are normal isotopies $T_i^j \to \mathcal{T}_i$, and we let c^j be the inverse image of c with the induced transverse orientation. Cut each T_i^j along c^j, and let the copy of c^j in the direction of the transverse orientation be c_+^j, and let c_-^j be the other copy of c^j. We then obtain an *infinite normal annulus* A by identifying c_+^j to c_-^{j+1} by normal isotopy for each j, and c_-^1 to c. We call the homotopy class of c the slope of A and c its boundary curve.

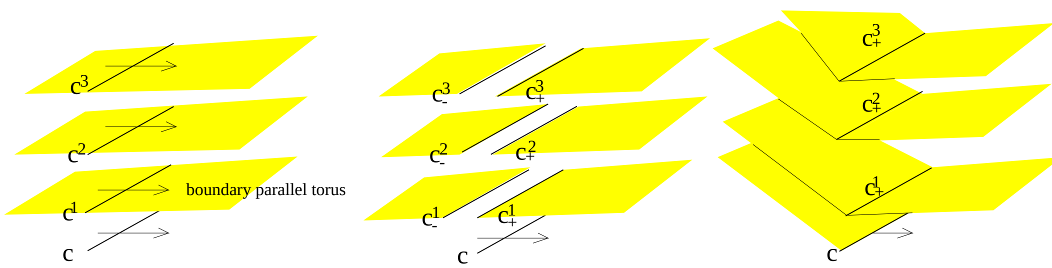


FIGURE 4.8. Constructing an infinite normal annulus

The transverse orientation of c determines the direction in which A *spirals into the cusp* — the other direction is obtained if in the above construction, we identify c_-^j to c_+^{j+1} by normal isotopy for each j. If two infinite normal annuli A_1 and A_2 with the same slope and disjoint boundary curves spiral into a cusp in different directions, then they intersect in an infinite collection of circles, and after performing regular exchanges, we obtain a finite annulus bounded by the boundary curves of A_1 and A_2, and an infinite family of boundary parallel tori.

4.1.10. Transverse orientations. Choose a compact core M' of M such that each boundary component T_i of M' is disjoint from S'. We denote the induced triangulation of T_i by $\mathcal{T}_i$. Each vertex v of $\mathcal{T}_i$ is contained in a unique edge e of $\mathcal{T}$. Let Δ be a 2–simplex of $\mathcal{T}_i$ containing v. Then there is an ideal vertex a of e such that the edges of Δ are a–type arcs on any face of any tetrahedron in $\mathcal{T}$ containing them. We say that v is *near* a.

Let Δ be a triangle in $\mathcal{T}_i$ and σ be the tetrahedron in $\mathcal{T}$ which contains Δ. Let s be a side of Δ, and F be the face of σ containing s. There is a unique quadrilateral type q in σ such that q and s have the same arc type on F, as shown in Figure 4.9. Let q be the *Q–modulus of s (with respect to Δ)*, and give s a transverse orientation (with respect to Δ) by attaching a little arrow pointing into the interior of Δ. Note that this construction is dual to the labelling of the vertices.

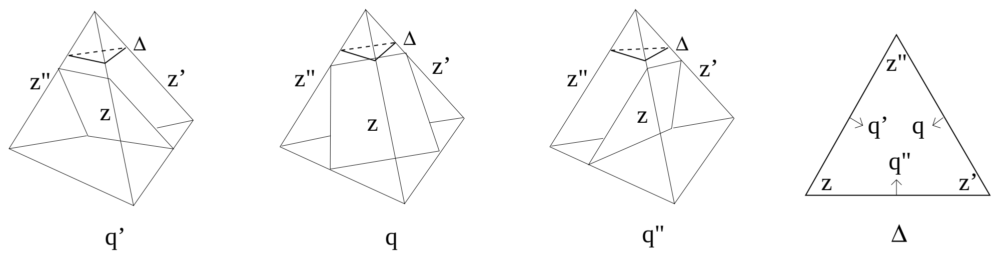
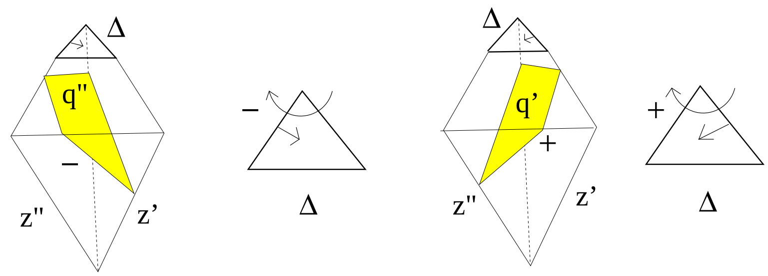


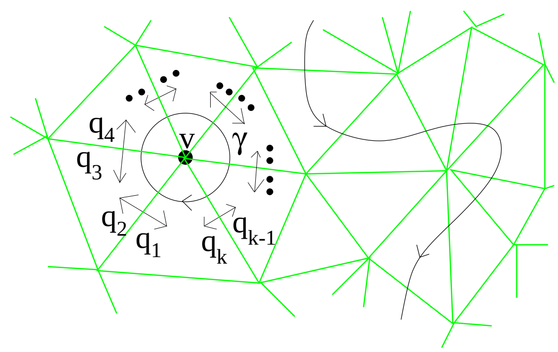
FIGURE 4.9. Q–moduli of triangle edges

To any closed oriented path γ in T_i which is disjoint from the 0–skeleton of $\mathcal{T}_i$ and meets the 1–skeleton transversely, we can associate a linear combination $\nu(\gamma)$ in the quadrilateral types by taking the positive Q–modulus of an edge if it crosses with the transverse orientation, and by taking the negative Q–modulus if it crosses against it (where each edge of $\mathcal{T}_i$ is counted twice — using the two adjacent triangles). If the vertex v of $\mathcal{T}_i$ is contained on the edge e in $\mathcal{T}$ and near a, we obtain the collection of 2–simplices meeting at v in $\mathcal{T}_i$ by truncating $B(e)$ at a. Moreover, from Figure 4.10, we deduce that the combination $\nu(\gamma)$ associated to a small circle with clockwise orientation around v is the Q–matching equation of e.

We can evaluate $\nu(\gamma)$ at the solution N to the Q-matching equations, giving an integer $\nu_N(\gamma)$. Moreover, since $\nu_N(\gamma) = 0$ if γ is a small circle about a vertex of $\mathcal{T}_i$, we conclude that ν_N is well-defined for homotopy classes of loops which intersect the 1–skeleton transversely away from the 0–skeleton, and hence defines a homomorphism $\nu_N : \pi_1(T_i) \to \mathbb{Z}$.



(a) Quadrilateral slopes and transverse orientation



(b) $\nu(\gamma) = \sum_{i=1}^{k}(-1)^i q_i$ is precisely the Q–matching equation of the edge containing v.

FIGURE 4.10. Q–moduli of edges in $\mathcal{T}_i$ and slopes of Q–discs

4.1.11. Simple closed curves on $\mathbf{T_i}$. Given an admissible solution N, we now associate a modulus and a transverse orientation to edges of $\mathcal{T}_i$ without reference to 2-simplices. Identify Q–moduli with the values given by N. Let s be a 1–simplex of $\mathcal{T}_i$ and let Δ and Δ' be the 2-simplices of $\mathcal{T}_i$ meeting in s, and n and n' be the associated Q–moduli. If $n = n'$, then give s the modulus zero and no transverse orientation. If $n > n'$, then give s the modulus $n - n'$ and the transverse orientation inherited from Δ, and if $n < n'$, then give s the modulus $n' - n$ and the transverse orientation inherited from Δ'. The modulus of s determines the number of unglued quadrilateral edges of the same arc type as s and the transverse orientation determines the tetrahedron in which the corresponding quadrilaterals are contained (compare Figures 4.12 and 4.13 to Figure 4.7).

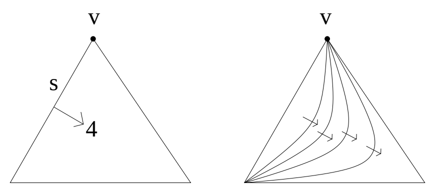


FIGURE 4.11. Construction and labelling of oriented arcs

Let c' be the closed simplicial curve consisting of edges in $\mathcal{T}_i$ with non–zero modulus. We now derive a unique collection of simple closed curves on T_i from c'. If s is a 1–simplex of c' with modulus n and transverse orientation pointing into Δ, then place n arcs in Δ, each of which has endpoints identical to s, any two of which only meet in their endpoints, and give all of them the induced transverse orientation (pointing towards the vertex of Δ opposite s; see Figure 4.11). Each of these arcs corresponds to an unglued edge of a quadrilateral.

We now identify endpoints of arcs at a vertex v of $\mathcal{T}_i$, such that the identifications are dual to the construction of "arc–adding pairs" on H_a. Please refer to Figures 4.12 and 4.13. Let H_v be the disc in $\mathcal{T}_i$ consisting of all triangles meeting at v, and let A be the set of arcs which we have not paired yet. An *outermost pair* γ and δ satisfies the following two criteria:

1. When we identify the endpoints of γ and δ at v, then the transverse orientations of γ and δ match.
2. The disc cut out from H_v by $\gamma \cup \delta$ which the transverse orientation of δ points away from does not contain any arcs in A.

If we have found an outermost pair, we identify their endpoints and isotope the union away from v. We then repeat the above by replacing A by $A - \{\gamma, \delta\}$.

Thus, after gluing and isotopy, we obtain a disjoint collection $\mathcal{C}$ of simple closed curves, each with a transverse orientation. Moreover, on may check that each $c \in \mathcal{C}$ is homotopic to a unique closed (possibly not simple) curve c'' of unglued quadrilateral edges.[1] We now describe how c determines a T–surface N'_c with $\partial N'_c = c''$ which will be attached to S' along c''. For details see [**31**].

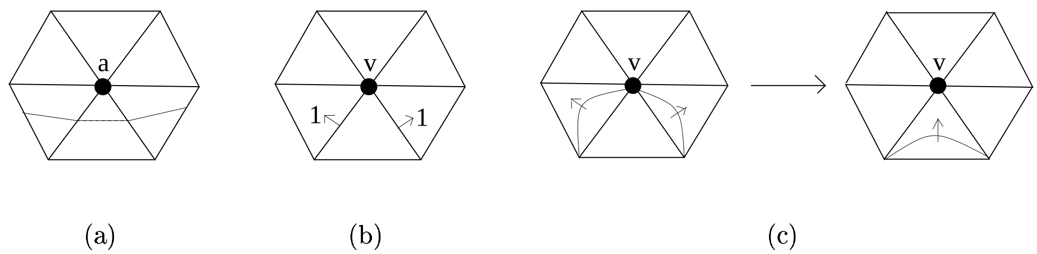


FIGURE 4.12. Example 1: (a) The pairing on H_a. (b) The edge labelling on $\mathcal{T}_i$. (c) Introducing arcs and resolving intersection points at v.

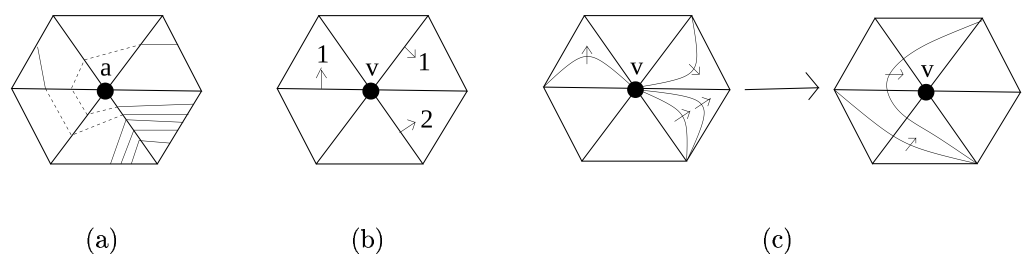


FIGURE 4.13. Example 2: (a) The pairing on H_a. (b) The edge labelling on $\mathcal{T}_i$. (c) Introducing arcs and resolving intersection points at v.

4.1.12. We first attach finite T–surfaces. Assume that $c \in \mathcal{C}$ is contractible, and hence bounds a disc D on T_i. If the transverse orientation of c points outside D, we let $N_c = D$, otherwise we let $N_c = \overline{T_i - D}$. The surface N'_c is obtained from N_c by a homotopy which takes c to c''. In the case $N_c = D$ it is easy to see that c'' has at most one point of self intersection, and hence the interior of N'_c is homeomorphic to the interior of a disc. The intersection of $S' \cup N'_c$ with

[1]This can e.g. be done by introducing labels for the arcs specifying a 1– and a 2–simplex of $\mathcal{T}_i$ and a quadrilateral $\square$.

each ideal tetrahedron is normal. We successively attach finite T–surfaces to S' for the contractible curves in $\mathcal{C}$, and perform regular exchanges between triangles if necessary. We denote the resulting 2–complex again by S', and we may assume that $S' \cap T_i = \emptyset$.

Hence assume that c is not contractible. In this case, we attach an infinite normal annulus A_c to S' along the corresponding curve c''. A_c is unique up to normal isotopy since we want the intersection of $S' \cup A_c$ with every tetrahedron to be normal. In particular, A_c is uniquely determined (up to normal isotopy) by c and its transverse orientation (see Subsection 4.1.9).

We successively attach normal annuli to S' for the essential curves in $\mathcal{C}$, perform regular exchanges between intersections and discard boundary parallel components that may occur. We thus obtain a normal surface S which is unique due to the construction. This completes the proof of Theorem 4.1.

Since annuli spiral into cusps in two different directions, and the direction is determined by the transverse orientation, we see that if there are two adjacent curves in $\mathcal{C}$ which have transverse orientations pointing away from each other, we can in fact attach a finite annulus to the corresponding curves in $\partial S'$. It follows inductively that if there are d essential curves in $\mathcal{C}$, d_1 of which have the same transverse orientation, then the intersection of the resulting surface S with a sufficiently small cusp cross section contains precisely $|d_1 - d_2|$ curves, where $d_2 = d - d_1$. This observation leads to the following:

PROPOSITION 4.2. *Let N be an admissible integer solution to the Q–matching equations, and denote the associated normal surface by S. For each torus component T_i of the boundary of M and for all $\gamma \in \text{im}(\pi_1(T_i) \to \pi_1(M))$, $|\nu_N(\gamma)|$ is the geometric intersection number of a loop representing γ in ∂M with ∂S.*

PROOF. Recall that c' is the closed simplicial curve consisting of edges in $\mathcal{T}_i$ with non–zero modulus. Let γ be an oriented closed path on T_i which meets the 1–skeleton of $\mathcal{T}_i$ transversely. We define $\nu_{c'}(\gamma)$ to be the sum of moduli over all

1–simplices of c' which γ crosses, taken with sign according to whether γ crosses with or against the transverse orientation. Thus, $\nu_{c'}(\gamma) = \nu_N(\gamma)$.

For each $c \in \mathcal{C}$, we can define a number $\nu_c(\gamma)$: If γ is a closed oriented curve on T_i which intersects c transversely, we associate the sign $+1$ to an intersection where γ crosses c with the transverse orientation, and -1 otherwise. The number $\nu_c(\gamma)$ is defined to be the sum of these signs, and is clearly well–defined for homotopy classes of c and γ. We have $\nu_N(\gamma) = \nu_{c'}(\gamma) = \sum_{c\in\mathcal{C}} \nu_c(\gamma)$.

Note that if $c \in \mathcal{C}$ is contractible, then $\nu_c(\gamma) = 0$. If c_0 is homotopic to c_1 with the opposite transverse orientation, then $\nu_{c_0}(\gamma) = -\nu_{c_1}(\gamma)$. Thus, if c is essential and there are d_1 curves in $\mathcal{C}$ which are homotopic to c with the same orientation, and d_2 curves which are homotopic to c with the opposite orientation, then $\nu_N(\gamma) = (d_1 - d_2)\nu_c(\gamma)$.

We also know that the number of parallel infinite annuli spiralling into the cusp is $|d_1 - d_2|$. The number $|\nu_c(\gamma)|$ clearly measures the geometric intersection number of c and γ, and this completes the proof of the proposition. ■

The argument in the proof of the above proposition gives precise information about the boundary curves of our normal surface:

COROLLARY 4.3. *Choose a set of generators* $\{\mathcal{M}_i, \mathcal{L}_i\}$ *for each boundary torus* T_i. *We may identify* $\pi_1(T_i)$ *with the first homology group of the torus and use additive notation. Under the assumptions of the previous proposition, we have*

1. *If* $\nu_N(\mathcal{M}_i) = \nu_N(\mathcal{L}_i) = 0$, *then* S *is disjoint from the* i*–th cusp.*[2]
2. *If* $\nu_N(\mathcal{M}_i) \neq 0$ *or* $\nu_N(\mathcal{L}_i) \neq 0$, *then let* $d > 0$ *denote the greatest common divisor of the numbers* $|\nu_N(\mathcal{M}_i)|$ *and* $|\nu_N(\mathcal{L}_i)|$. *Put* $p = -\nu_N(\mathcal{L}_i)/d$ *and* $q = \nu_N(\mathcal{M}_i)/d$. *Then* $s = p\mathcal{M}_i + q\mathcal{L}_i$ *has* $\nu_N(s) = 0$ *and hence it is a boundary slope of* S. *Furthermore,* S *has* d *boundary components on* T_i.

[2]This is also shown in the file "*normal_surface_construction.c*" of [**50**].

The (unoriented) boundary curves of an embedded normal surface S with normal Q–coordinate $N = N(S)$ are therefore determined by the vector:

$$(-\nu_N(\mathcal{L}_1), \nu_N(\mathcal{M}_1), \ldots, -\nu_N(\mathcal{L}_h), \nu_N(\mathcal{M}_h)) \in \mathbb{Z}^{2h}. \tag{4.2}$$

Note that this is not necessarily a point of the boundary curve space $\mathfrak{BC}(M)$, since S may not be essential. Also note that we can associate projectivised boundary curves to projective classes of normal surfaces, and hence, for each $N \in P(M)$, we obtain a point

$$[-\nu_N(\mathcal{L}_1), \nu_N(\mathcal{M}_1), \ldots, -\nu_N(\mathcal{L}_h), \nu_N(\mathcal{M}_h)] \in \mathbb{R}P^{2h-1}/\mathbb{Z}_2^{h-1}. \tag{4.3}$$

Note that if N and N' are admissible solutions such that $N+N'$ is admissible, then $\nu_N + \nu_{N'} = \nu_{N+N'}$, and we can determine boundary curves of linear combinations using the vectors of the form (4.2).

Remark: Craig Hodgson has recently observed that the approach of Weeks sketched in Subsection 4.1.6 can be used to simplify the proofs of Proposition 4.2 and Corollary 4.3.

4.2. Ideal points and normal surfaces

As in the previous section, we assume that M is the interior of a compact, orientable 3–manifold with non–empty boundary consisting of a disjoint union of tori, fix any topological ideal triangulation $\mathcal{T}$ of M, and let n be the number of tetrahedra in $\mathcal{T}$.

Let $\pi_+ = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. We define a variety $\mathfrak{D}(M)$ in $(\mathbb{C} - \{0\})^{3n}$ with the property that each point of $\mathfrak{D}(M) \cap \pi_+^{3n}$ defines a (possibly incomplete) hyperbolic structure on M, and the logarithmic limit set of $\mathfrak{D}(M)$ can be identified with a compact subset in the projective admissible solution space $\mathcal{N}(M)$.

4.2.1. Shape parameters. An *ideal hyperbolic tetrahedron* is an oriented geodesic ideal tetrahedron in hyperbolic 3–space with all vertices on the sphere at infinity. An *ideal hyperbolic triangulation* of M is an ideal triangulation of M in which all ideal 3–simplices are hyperbolic, and the gluing is realised by isometries of $\mathbb{H}^3$. As before, we will often drop the adjective "ideal".

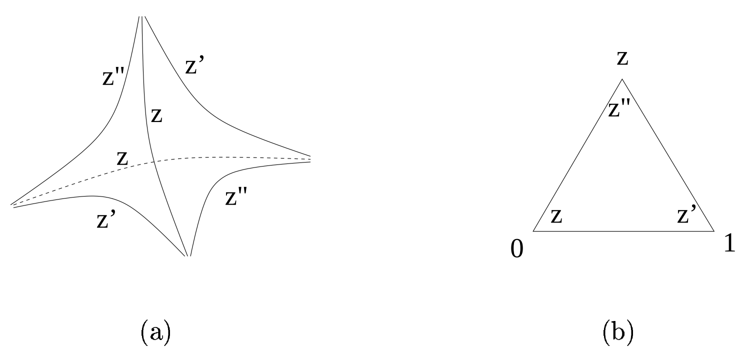


FIGURE 4.14. Edge invariants: (a) Ideal hyperbolic tetrahedron. (b) Link of a vertex (regarded as a triangle in $\mathbb{C}$)

We now paraphrase material of Thurston [**44**], and Neumann and Zagier [**37**]. An ideal hyperbolic tetrahedron σ_h is (up to orientation preserving isometry) uniquely determined by edge invariants, which we call *shape parameters*. Shape parameters are complex numbers in $\mathbb{C} - \{0, 1\}$, and arise from the fact that the intersection of σ_h with a small horosphere based at one of its vertices is a Euclidean triangle whose similarity class determines σ_h. Following [**44**], we view this triangle from the ideal vertex, which is consistent with the convention in Section 4.1. Note that in [**37**] the triangle is viewed from the manifold.

A geometric argument shows that opposite edges of σ_h have the same parameter, and if the labelling is as in Figure 4.14, then the three invariants satisfy:

$$(p) \qquad 1 = z(1 - z''),$$

$$(p') \qquad 1 = z'(1 - z),$$

$$(p'') \qquad 1 = z''(1 - z').$$

Any ideal hyperbolic tetrahedron determines a triple (z, z', z'') satisfying the above equations, and conversely, any solution (z, z', z'') to the above equations determines an ideal hyperbolic tetrahedron up to orientation preserving isometry. Note that any edge invariant uniquely determines the other two. For $z \in \mathbb{C} - \{0, 1\}$, we have:

$$z' = \frac{1}{1-z} \quad \text{and} \quad z'' = 1 - \frac{1}{z}. \tag{4.4}$$

An ideal hyperbolic tetrahedron is *positively oriented* if $\Im(z) > 0$, *flat* if $z \in \mathbb{R} - \{0, 1\}$, and *negatively oriented* if $\Im(z) < 0$. This terminology is independent of the chosen edge invariant.

We can *deform* an ideal hyperbolic tetrahedron by varying its shape parameters smoothly. We say that a hyperbolic tetrahedron *degenerates* when it is deformed such that one of its shape parameters converges to zero, one or infinity on the Riemann sphere. When a tetrahedron degenerates, the triple (z, z', z'') approaches a cyclic permutation of the triple $(0, 1, \infty)$. An ideal hyperbolic triangulation is said to *degenerate* if it is deformed so that at least one of its ideal simplices degenerates.

4.2.2. Deformation variety. Consider the topological ideal triangulation $\mathcal{T}$ of M. Let the ideal tetrahedra be $\{\sigma_1, \ldots, \sigma_n\}$, and recall that there are exactly n edges $e_1, \ldots, e_n$ in $\mathcal{T}$ since $\chi(M) = 0$. As in Subsection 4.1.2, we assign edge labels z_i, z_i', z_i'' to σ_i, and now think of these labels as complex parameters. Thus, we can give σ_i a hyperbolic structure if and only if its edge labels satisfy the relations:

$$z_i(1 - z_i'') = 1, \tag{p_i}$$

$$z_i'(1 - z_i) = 1, \tag{p_i'}$$

$$z_i''(1 - z_i') = 1. \tag{p_i''}$$

For $Z = (z_1, z_1', z_1'', \ldots, z_n'')$ subject to the system of equations $\{p_i, p_i', p_i''\}_{i=1,\ldots,n}$, let $\mathcal{T}_Z$ be the triangulation where we put the structure of an ideal hyperbolic tetrahedron on each σ_i as specified by the edge invariants, and glue the tetrahedra abstractly (face to face) as in $\mathcal{T}$.

The hyperbolic structures on the tetrahedra in $\mathcal{T}_Z$ fit together to give a (possibly incomplete) hyperbolic structure on M if $Z \in \pi_+^{3n}$ and the tetrahedra abutting an edge should "close up", as Neumann and Zagier put it. Algebraically speaking, we require that the sum of dihedral angles at each edge e_j is equal to 2π, and that the following *hyperbolic gluing equation* is satisfied:

$$1 = \prod_{i=1}^{n} z_i^{a_{ij}} (z_i')^{a_{ij}'} (z_i'')^{a_{ij}''}, \tag{4.5}$$

where, as in Subsection 4.1.5, $a_{ij}^{(k)}$ is the number of times e_j has label $z_i^{(k)}$. The exponents are therefore contained in the set $\{0, 1, 2\}$.

We represent parameter relations as elements in $\mathbb{C}[z_1^{\pm 1}, \ldots, (z_n'')^{\pm 1}]$ by:

$$p_i = z_i(1 - z_i'') - 1, \quad p_i' = z_i'(1 - z_i) - 1, \quad p_i'' = z_i''(1 - z_i') - 1, \tag{4.6}$$

and similarly for the hyperbolic gluing equations we put:

$$g_j = \prod_{i=1}^{n} z_i^{a_{ij}} (z_i')^{a_{ij}'} (z_i'')^{a_{ij}''} - 1. \tag{4.7}$$

The *deformation variety* $\mathfrak{D}(M)$ is the variety in $(\mathbb{C} - \{0\})^{3n}$ defined by the hyperbolic gluing equations together with the parameter relations. The following result is probably well known, though it is worth pointing out, since we do not assume that there is a point $Z_0 \in \mathfrak{D}(M)$ corresponding to a complete hyperbolic structure on M, and we also do not require solutions to be near such a point.

PROPOSITION 4.4. *Let M be the interior of a compact, orientable 3–manifold with non–empty boundary consisting of a disjoint union of tori, and let $\mathcal{T}$ be a topological ideal triangulation of M. For each $Z \in \mathfrak{D}(M) \cap \pi_+^{3n}$, M has a (possibly incomplete) hyperbolic structure, such that the topological ideal triangulation $\mathcal{T}$ is isotopic to a hyperbolic ideal triangulation.*

PROOF. Choose disjoint isometric embeddings of the tetrahedra in $\mathcal{T}_Z$ into $\mathbb{H}^3$, and realise the face pairings by restrictions of isometries of $\mathbb{H}^3$. The resulting identification space will be homeomorphic to M. Moreover, if the angles around each edge add up to exactly 2π, then we obtain a (possibly incomplete) hyperbolic structure on M, and viewing both $\mathcal{T}$ and $\mathcal{T}_Z$ as subsets of M, we can isotope one to the other.

If the argument of a complex number is chosen in $(0, 2\pi]$, then

$$\pi = \arg(z) + \arg(z') + \arg(z'') \qquad \text{if } \Im(z) > 0. \tag{4.8}$$

Moreover, the argument of each edge parameter is the dihedral angle along the edge.

The hyperbolic gluing equation of the edge e_j:

$$1 = \prod_{i=1}^{n} z_i^{a_{ij}} (z_i')^{a_{ij}'} (z_i'')^{a_{ij}''}$$

implies that the sum of arguments satisfies:

$$2\pi n_j = \sum_{i=1}^{n} \big(a_{ij} \arg(z_i) + a_{ij}' \arg(z_i') + a_{ij}'' \arg(z_i'')\big),$$

for some integer $n_j > 0$. Recalling that the number of edges equals the number of tetrahedra, summing over all edges gives:

$$\sum_{j=1}^{n} 2\pi n_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \big(a_{ij} \arg(z_i) + a_{ij}' \arg(z_i') + a_{ij}'' \arg(z_i'')\big).$$

Since the argument of each shape parameter appears in the above sum exactly twice, the equation simplifies to:

$$2\pi \sum_{j=1}^{n} n_j = 2 \sum_{i=1}^{n} \big(\arg(z_i) + \arg(z_i') + \arg(z_i'')\big) = 2\pi n,$$

since by assumption all tetrahedra are positively oriented and the arguments are chosen such that they add up to π for each tetrahedron. But each n_j is greater or equal to one, and hence each n_j must be equal to exactly 1. Thus, the sum of angles around each edge is equal to 2π, and this concludes the proof. ∎

By symmetry, the above can be modified for $Z \in \mathfrak{D}(M)$ with only negatively oriented tetrahedra. There is an obvious relationship between these solutions, since if $Z = (z_1, \dots, z''_n) \in \mathfrak{D}(M)$, then $\overline{Z} = (\overline{z}_1, \dots, \overline{z}''_n) \in \mathfrak{D}(M)$, and the corresponding hyperbolic structures are related by an orientation reversing isometry.

4.2.3. Since $\mathfrak{D}(M)$ is a variety in $(\mathbb{C} - \{0\})^{3n}$, we can use the logarithmic limit set to define its set of ideal points. We say that $\mathcal{T}$ *degenerates*, if there is a sequence $\{Z_i\} \subset \mathfrak{D}(M)$, such that at least one of the tetrahedra in $\mathcal{T}_{Z_i}$ degenerates as $i \to \infty$.

The deformation variety is suited for the study of denenerating triangulations, since whenever an edge invariant of a tetrahedron converges to one, the other two edge invariants "blow up". Thus, an ideal point of $\mathfrak{D}(M)$ is approached if and only if a tetrahedron degenerates. The defining equations of the deformation variety keep the symmetry between edges of tetrahedra and together with our description of normal surfaces, the set–up will lead us to associate a point in the projectivised admissible solution space $\mathcal{N}(M)$ to each ideal point of $\mathfrak{D}(M)$.

4.2.4. Recall the description of the hyperbolic gluing equation (4.5) of e_j:

$$1 = \prod_{i=1}^{n} z_i^{a_{ij}} (z'_i)^{a'_{ij}} (z''_i)^{a''_{ij}},$$

and the Q–matching equation (4.1) of e_j:

$$0 = \sum_{i=1}^{n} (a''_{ij} - a'_{ij}) q_i + (a_{ij} - a''_{ij}) q'_i + (a'_{ij} - a_{ij}) q''_i.$$

To state the relationship between these sets of equations as a matrix equation, let

$$A = \begin{pmatrix} a_{11} & a'_{11} & a''_{11} & a_{21} & \dots & a''_{n1} \\ \vdots & & & & & \vdots \\ a_{1n} & \dots & & & & a''_{nn} \end{pmatrix}, \tag{4.9}$$

be the matrix containing the exponents of the hyperbolic gluing equations, and let

$$B = \begin{pmatrix} a''_{11} - a'_{11} & a_{11} - a''_{11} & a'_{11} - a_{11} & \dots & a'_{n1} - a_{n1} \\ \vdots & & & & \vdots \\ a''_{1n} - a'_{1n} & \dots & & & a'_{nn} - a_{nn} \end{pmatrix}, \tag{4.10}$$

be the matrix containinig the coefficients of the Q–matching equations. Let

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{4.11}$$

and let C_n be the $(3n \times 3n)$ block diagonal matrix with n copies of C on its diagonal. Then $C_n = -C_n^T$, and

$$AC_n = B. \tag{4.12}$$

4.2.5. Tentative logarithmic limit set. The deformation variety $\mathfrak{D}(M)$ is not defined by a principal ideal, hence its logarithmic limit set $\mathfrak{D}_\infty(M)$ is in general not directly determined by its defining equations. However, it is contained in the intersection of the spherical duals of its defining equations:

$$\mathfrak{D}_\infty(M) \subseteq \mathfrak{D}_?(M) = \bigcap_{i=1}^{n} \big(Sph(g_i) \cap Sph(p_i) \cap Sph(p_i') \cap Sph(p_i'')\big). \tag{4.13}$$

We call $\mathfrak{D}_?(M)$ the *tentative logarithmic limit set* of M. In order to give a description of this set, consider first the intersection:

$$S_n = \bigcap_{i=1}^{n} \big(Sph(p_i) \cap Sph(p_i') \cap Sph(p_i'')\big). \tag{4.14}$$

A calculation using the equations of Subsection 3.1.5 shows that each point in S_n is made up of n coordinate triples, each of the form $(0, x, -x)$, $(-x, 0, x)$ or $(x, -x, 0)$, where $x \geq 0$. (If $x > 0$ these correspond to the cases where $z' \to \infty$, $z'' \to \infty$ or $z \to \infty$ respectively.) Similarly, one obtains that each hyperbolic gluing equation gives rise to the intersection of S_n with a hyperplane, leaving all $\xi \in S_n$ such that

$$(a_{1j}, a_{1j}', \ldots, a_{nj}'')^T \cdot \xi = 0. \tag{4.15}$$

Thus, $\mathfrak{D}_?(M)$ is the intersection of S_n with the nullspace of A.

PROPOSITION 4.5. *Let M be the interior of an orientable compact 3–manifold with non–empty boundary consisting of tori, and $\mathcal{T}$ be an ideal triangulation of*

M. The tentative logarithmic limit set $\mathfrak{D}_?(M)$ is homeomorphic to the projective admissible solution space $\mathcal{N}(M)$.

PROOF. We have defined $\mathcal{N}(M)$ as collection of elements in the nullspace of B with the property that at most one quadrilateral type has non–zero coordinate for each tetrahedron, all coordinates are ≥ 0 and their sum is equal to 1. We may project this set from the unit simplex onto the sphere of radius $1/\sqrt{2}$ centered at the origin in $\mathbb{R}^{3n}$, and, for simplicity, we denote this set again by $\mathcal{N}(M)$.

The map C_n^T takes $\mathcal{N}(M)$ to the unit sphere S^{3n-1}, where $\mathfrak{D}_?(M)$ is found, since we have the following correspondence between the i–th coordinate triples:

$$\begin{pmatrix} 0 \\ x \\ -x \end{pmatrix} = C^T \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -x \\ 0 \\ x \end{pmatrix} = C^T \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} = C^T \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}. \tag{4.16}$$

Thus, if $z \in \mathcal{N}(M)$ satisfies $||z||^2 = 1/2$, then $||C_n^T z||^2 = 2||z||^2 = 1$. Furthermore, given $z \in \mathcal{N}(M)$, we see that $C_n^T z \in S_n$, the set which contains $\mathfrak{D}_?(M)$. Now $0 = Bz = A(C_n z) = -A(C_n^T z)$ implies that $C_n^T z \in \mathfrak{D}_?(M)$. We therefore have a linear map $\mathcal{N}(M) \to \mathfrak{D}_?(M)$, and we claim that it is 1–1 and onto.

The kernel of C_n^T is generated by the vectors with $(1,1,1)^T$ in the i–th triple and 0 in the other positions. It follows that different admissible solutions cannot differ by an element in the kernel. The map is therefore 1–1.

Since every element in S_n has a unique inverse image under C_n^T, we can take any $\xi \in \mathfrak{D}_?(M)$ to a normal Q–coordinate $N(\xi)$ using (4.16). Thus, $\xi = C_n^T N(\xi)$, and hence $BN(\xi) = 0$. This shows that the map is onto, and in fact, that we have a well–defined inverse mapping. ■

Thus, any degeneration of an ideal triangulation corresponds to a unique normal surface. However, the example of the Whitehead link in Chapter 5 shows that not all normal surfaces arise in this way.

4.2.6. For $\xi \in \mathfrak{D}_\infty(M)$, we let $N(\xi)$ be the unique normal Q–coordinate such that $\xi = C_n^T N(\xi)$ as in the proof of Proposition 4.5. If ξ has rational coordinate ratios, we let $S(\xi)$ be the minimal representative of the projective class defined by $N(\xi)$. The properties we study in the following are independent of the choice of $S(\xi)$, but it will be convenient to refer to a surface, rather than a solution.

4.3. Holonomy variety and essential surfaces

In this section, we relate the derivative of the holonomy of [**37**] to the function $\nu(\gamma)$ of Subsection 4.1.10, and use this to prove the following:

PROPOSITION 4.6. *Let M be the interior of an orientable compact 3–manifold with non–empty boundary consisting of tori, and let $\mathcal{T}$ be an ideal triangulation of M.*

Let $\xi \in \mathfrak{D}_\infty(M)$ be an ideal point with rational coordinate ratios, and assume that the normal surface $S(\xi)$ is not closed. Then there is an essential surface in M which has (up to projectivisation) the same boundary curves as $S(\xi)$.

4.3.1. Let M be the interior of an orientable compact 3–manifold with non–empty boundary consisting of tori, and $\mathcal{T}$ be an ideal triangulation of M. Let n be the number of ideal tetrahedra in $\mathcal{T}$, and h be the number of vertices (i.e. cusps of M). We give the universal cover $\tilde{M}$ the ideal triangulation induced by $\mathcal{T}$, such that the covering projection $p : \tilde{M} \to M$ is simplicial. For each $Z \in \mathfrak{D}(M)$, we now describe a map $\Phi_Z : \tilde{M} \to \mathbb{H}^3$ following [**51**]. Each ideal tetrahedron in $\tilde{M}$ inherits edge parameters from Z. Choose a tetrahedron σ in $\tilde{M}$ and an embedding of σ into $\mathbb{H}^3$ of the specified shape. For each tetrahedron of $\tilde{M}$ which has a face in common with σ, there is a unique embedding into $\mathbb{H}^3$ which coincides with the embedding of σ on the common face and which has the shape determined by Z. Thus, starting with an embedding of $\sigma \subset \tilde{M}$, there is a unique way to extend this to a continuous map $\Phi_Z : \tilde{M} \to \mathbb{H}^3$ such that each tetrahedron in $\tilde{M}$ is mapped to a hyperbolic tetrahedron of the specified shape. If $Z \in \pi_+^{3n}$, it follows from Proposition 4.4 that this is a developing map for the hyperbolic structure on M, but if some tetrahedra are flat or negatively oriented, it is still well–defined.

For each $Z \in \mathfrak{D}(M)$, Φ_Z can be used to define a representation $\overline{\rho}_Z : \pi_1(M) \to PSL_2(\mathbb{C})$ (see [**51**]). If we think of a representation into $PSL_2(\mathbb{C})$ as an action on $\mathbb{H}^3$, then this is the unique representation which makes Φ_Z $\pi_1(M)$–equivariant: $\Phi_Z(\gamma \cdot x) = \overline{\rho}(\gamma)\Phi_Z(x)$ for all $x \in tildeM$, $\gamma \in \pi_1(M)$. Thus, $\overline{\rho}_Z$ is well–defined up to conjugation, since it only depends upon the choice of the embedding of the initial tetrahedron σ.

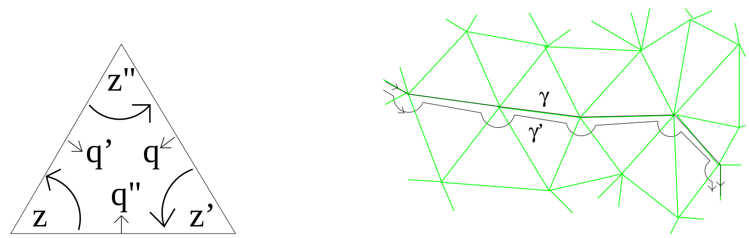
4.3.2. Holonomies. Let M' be the compact core of M, and denote its boundary tori by $T_1, \ldots, T_h$, and give each of them the triangulation $\mathcal{T}_i$ induced by $\mathcal{T}$. Let γ be a closed simplicial path on T_i. In [**37**], the *holonomy* $\mu(\gamma)$ is defined as $(-1)^{|\gamma|}$ times the product of the moduli z for the triangle vertices touching γ on the right, where $|\gamma|$ is the number of 1–simplices of γ, and the moduli are as defined in Subsection 4.1.2.

At $Z \in \mathfrak{D}(M)$, we can evaluate $\mu(\gamma)$, giving a complex number $\mu_Z(\gamma) \in \mathbb{C} - \{0\}$. It follows from [**44, 37**], that $(\operatorname{tr} \overline{\rho}_Z(\gamma))^2 = \mu_Z(\gamma) + 2 + \mu_Z(\gamma)^{-1}$, and hence that $\mu_Z(\gamma)$ is the square of an eigenvalue. Choose a basis $\{\mathcal{M}_i, \mathcal{L}_i\}$ for each boundary torus T_i, and give the eigenvalue variety $\overline{\mathfrak{E}}(M)$ the corresponding coordinates $(M_1, L_1, \ldots, M_h, L_h)$. Since $\mu_Z : \pi_1(T_i) \to \mathbb{C} - \{0\}$ is a homomorphism for each $i = 1, \ldots, h$, we obtain a well–defined map:

$$\overline{\mathrm{e}} : \mathfrak{D}(M) \to \overline{\mathfrak{E}}(M) \qquad \overline{\mathrm{e}}(z_1, \ldots, z_n'') = (\mu(\mathcal{M}_1), \ldots, \mu(\mathcal{L}_h)).$$

The closure of the image is called the *holonomy variety* of M. In the 1–cusped case, some work has been done on this variety and its defining equation in an Undergraduate Research Project at Columbia University [**8**], and by Bueti [**7**].

4.3.3. Simplicial version of $\nu(\gamma)$. For the purpose of this section, it will be convenient to have a simplicial definition of the function $\nu(\gamma)$ defined in Subsection 4.1.10. Let Δ be a triangle in the induced triangulation $\mathcal{T}_i$ of T_i. If v, u, t are the vertices of Δ in clockwise ordering (as viewed from the cusp), and q_0 is the Q–modulus of $[v,t]$, and q_1 is the Q–modulus of $[v,u]$, then define the *Q–modulus* of v to be $q_0 - q_1$ (with respect to Δ). E.g. if Δ is the triangle of Figure 4.15, then the Q–modulus of the vertex with label z is $q'' - q'$ with respect to Δ. Note that for a fixed vertex, the sum of moduli with respect to all triangles containing it is equal to the corresponding Q–matching equation.



(a) μ and ν (b) $\nu^{simp}(\gamma) = \nu(\gamma')$

FIGURE 4.15. μ and ν

If γ is an oriented simplicial path, let $\nu^{simp}(\gamma)$ be the sum of Q–moduli of vertices of triangles touching γ to the right. If γ' is the right hand boundary component of a small regular neighbourhood of γ, oriented in the same way as γ, then $\nu(\gamma') = \nu^{simp}(\gamma)$. We therefore write $\nu^{simp} = \nu$. Let $\nu_N(\gamma)$ be the evaluation of $\nu(\gamma)$ at N.

LEMMA 4.7. *Let N be a solution to the Q–matching equations. The number $\nu_N(\gamma) \in \mathbb{R}$ depends only on the homotopy class of γ and defines a homomorphism $\nu_N : \pi_1(T_i) \to (\mathbb{R}, +)$.*

PROOF. Any deformation of the path γ within its homotopy class is realised by successively applying elementary steps as illustrated in Figure 4.16 (cf. [**37**]). Denote the path after one such elementary step by γ'. The quadrilateral types which γ and γ' do not have in common are precisely the types around a vertex v. Note that in γ the occurring quadrilateral types around v have their sign associated

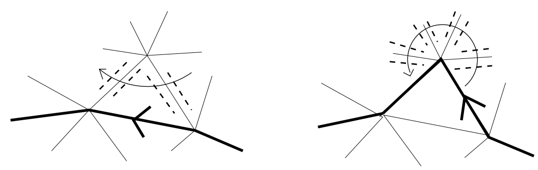


FIGURE 4.16. Homotopy of a simplicial path

as though we are travelling in clockwise order around v, whilst for γ', we travel in anticlockwise order. Thus, they differ by exactly the sum of all moduli of v, which is zero. ■

Let $\overrightarrow{\nu(\gamma)}$ be the coefficient (row) vector of $\nu(\gamma)$, and $\overrightarrow{\mu(\gamma)}$ be the exponent (row) vector of $\mu(\gamma)$, where $(q_1, q_1', q_1'', \ldots, q_n'')$ and $(z_1, z_1', z_1'', \ldots, z_n'')$ are the respective coordinate systems. Then:

LEMMA 4.8. $\overrightarrow{\nu(\gamma)} = \overrightarrow{\mu(\gamma)} C_n^T$.

PROOF. It it sufficient to verify the relationship for vertex moduli, and hence for the vertex labels of the triangle in Figure 4.15. If γ touches the vertex with label z to the right, then the contribution to $\mu(\gamma)$ is z, and the contribution to $\nu(\gamma)$ is $q'' - q'$. Similarly, z' corresponds to $q - q''$ and z'' to $q' - q$.

Describing this relationship for coordinate triples using C gives: $(1,0,0)C^T = (0,-1,1)$, $(0,1,0)C^T = (1,0,-1)$, and $(0,0,1)C^T = (-1,1,0)$. This proves the lemma. ■

4.3.4. Proof of Proposition 4.6. We fix a basis $\{\mathcal{M}_i, \mathcal{L}_i\}$ for each boundary torus T_i of M, and denote the resulting coordinates for $\overline{\mathfrak{E}}(M)$ by $(M_1, L_1, \ldots, M_h, L_h)$.

Let $\xi \in \mathfrak{D}_\infty(M)$ be an ideal point with rational coordinate ratios. Lemma 3.6 provides us with a curve C in $\mathfrak{D}(M)$ such that $\xi \in C_\infty$. Moreover, there is $\alpha > 0$ such that $\alpha\xi$ defines a normalised, discrete, rank 1 valuation v_ξ on $\mathbb{C}(C)$:

$$\alpha\xi = (-v_\xi(z_1), \ldots, -v_\xi(z_n'')).$$

Indeed, we may choose $\alpha > 0$ such that the above is an integer valued vector whose entries have no common divisor. Then the normal Q–coordinate of $S(\xi)$ is $\alpha N(\xi)$. We therefore denote $\alpha\xi$ and $\alpha N(\xi)$ by ξ and $N(\xi)$ respectively.

For each boundary torus T, and each $\gamma \in \operatorname{im}(\pi_1(T) \to \pi_1(M))$, there are $g_1, \ldots, g''_n \in \mathbb{Z}$ such that:

$$\mu(\gamma) = \prod_{i=1}^{n} z_i^{g_i}(z'_i)^{g'_i}(z''_i)^{g''_i}, \text{ and } \overrightarrow{\mu(\gamma)} = (g_1, \ldots, g''_n) \in \mathbb{Z}^{3n}.$$

Recall that we have $\xi = C_n^T N(\xi)$ and $C_n = -C_n^T$. Using this and Lemma 4.8, we get:

$$\begin{aligned} v_\xi(\mu(\gamma)) &= v_\xi\left(\prod_{i=1}^{n} z_i^{g_i}(z'_i)^{g'_i}(z''_i)^{g''_i}\right) = \sum_{i=1}^{n} g_i v_\xi(z_i) + g'_i v_\xi(z'_i) + g''_i v_\xi(z''_i) \\ &= -\overrightarrow{\mu(\gamma)} \cdot \xi = -\overrightarrow{\mu(\gamma)} \cdot C_n^T N(\xi) = -\overrightarrow{\nu(\gamma)} \cdot N(\xi) \\ &= -\nu_{N(\xi)}(\gamma) = \nu_{N(\xi)}(\gamma^{-1}) \end{aligned}$$

Thus, restricted to the boundary, we have $v_\xi\mu = -\nu_{N(\xi)}$, and in particular, since $S(\xi)$ is not closed, the eigenvalue of at least one peripheral element blows up. This implies that the restriction $\overline{\mathrm{e}} : C \to \overline{\mathfrak{E}}(M)$ is not constant, and its image is therefore a curve $C' \subset \overline{\mathfrak{E}}(M)$. Denote the ideal point of C' corresponding to ξ by ξ'. We obtain a corresponding normalised, discrete, rank 1 valuation v' of $\mathbb{C}(C')$ at ξ' as follows. If

$$(v_\xi\mu(\mathcal{M}_1), v_\xi\mu(\mathcal{L}_1), \ldots, v_\xi\mu(\mathcal{M}_h), v_\xi\mu(\mathcal{L}_h)) \tag{4.17}$$

contains a pair of coprime integers, define $v'(M_i) = v_\xi\mu(\mathcal{M}_i)$, and $v'(L_i) = v_\xi\mu(\mathcal{L}_i)$. Otherwise, let d denote the greatest common divisor of the entries in (4.17), and then define $v'(M_i) = \frac{1}{d}v_\xi\mu(\mathcal{M}_i)$, and $v'(L_i) = \frac{1}{d}v_\xi\mu(\mathcal{L}_i)$. In either case, we obtain a valuation of $\mathbb{C}(C')$ with the desired properties.

We can apply Culler-Shalen theory using ξ' and v'. Lemma 3.12 yields that the projectivised boundary curves of an essential dual surface are given by:

$$\begin{aligned}
&[v'(\mathcal{L}_1), -v'(\mathcal{M}_1), \ldots, v'(\mathcal{L}_h), -v'(\mathcal{M}_h)] \\
=&[v_\xi(\mu\mathcal{L}_1), -v_\xi(\mu\mathcal{M}_1), \ldots, v_\xi(\mu\mathcal{L}_h), -v_\xi(\mu\mathcal{M}_h)] \\
=&[-v_{N(\xi)}(\mathcal{L}_1), v_{N(\xi)}(\mathcal{M}_1), \ldots, -v_{N(\xi)}(\mathcal{L}_h), v_{N(\xi)}(\mathcal{M}_h)].
\end{aligned}$$

This proves the claim since the latter gives the projectivised boundary curves of $S(\xi)$ according to (4.3). ■

4.4. Hyperbolic manifolds

In this section we will assume that M admits a complete cusped hyperbolic structure of finite volume, and that the ideal triangulation $\mathcal{T}$ is chosen such that there is a point $Z_0 \in \mathfrak{D}(M) \cap \pi_+^{3n}$ which corresponds to the complete structure. In this case, we say that $\mathcal{T}$ *realises* the complete structure. The results of this section are direct consequences of [**37**].

LEMMA 4.9. *Let M be a complete orientable hyperbolic 3–manifold of finite volume with h cusps, and assume that $\mathcal{T}$ is a triangulation realising the complete structure. Let B be the coefficient matrix of the Q–matching equations, and n be the number of ideal tetrahedra in $\mathcal{T}$. Then the rank of B is $r(B) = n - h$. Moreover, we may choose $n - h$ rows of B such that every other row is an integer linear combination of these rows.*

PROOF. Note that substituting the expressions (4.4) for z' and z'' in terms of z into the hyperbolic gluing equations gives:

$$\begin{aligned}
1 &= \prod_{i=1}^{n} z_i^{a_{ij}} (z_i')^{a'_{ij}} (z_i'')^{a''_{ij}} \\
&= \prod_{i=1}^{n} z_i^{a_{ij}} \Big(\frac{1}{1-z_i}\Big)^{a'_{ij}} \Big(\frac{z_i - 1}{z_i}\Big)^{a''_{ij}} = \prod_{i=1}^{n} (-1)^{a''_{ij}} z_i^{a_{ij} - a''_{ij}} (1 - z_i)^{a''_{ij} - a'_{ij}}.
\end{aligned}$$

The matrix R defined in [**37**] contains the coefficients of z_i and $(1-z_i)$ from the above equations. Moreover, the columns of R are columns of B, and the remaining columns of B are linear combinations of the columns in R. It follows that the rank of B is equal to the rank of R, which under the hypothesis of this lemma is shown in [**37**] to be $n-h$. (The proof of this fact uses Mostow rigidity and $Z_0 \in \mathfrak{D}(M) \cap \pi_+^{3n}$. Without these assumptions, one only knows that $r(R) \leq n-h$.)

The proof of the second part is already contained in [**37**], as well. Let $X = (x_{ij})_{1\leq i\leq h, 1\leq j\leq n}$ such that x_{ij} is the number of vertices e_j has on the end of M corresponding to T_i. It is shown in [**37**] that X has rank h and that $XR = 0$. Thus, we also have $XB = 0$.

Each column of X contains either the entry 1 twice and zero in the remaining positions, or the entry 2 once and zero elsewhere. Denote the rows of B by $B_1, \ldots, B_n$. We obtain h linear combinations $X_i = \sum_{j=n}^{n} x_{ij} B_j$ for $i = 1, \ldots, h$. If $x_{ij} = 2$, then B_j only occurs in X_i with non–zero coefficient. Thus, if for some i, all coefficients $x_{ij} \in \{0, 2\}$, then the rows with non–zero coefficient only occur in X_i. We then divide X_i by 2 and solve for any occuring row without affecting the other linear combinations.

Now assume that for some $j = j_0$, we have $x_{ij_0} = 1$. Then there is exactly one $k \neq i$ such that $x_{kj_0} = 1$. We may solve X_i for B_{j_0}, and substituting into X_k gives $0 = X_k - X_i$, where all coefficients are contained in $\{0, \pm 1, \pm 2\}$. Moreover, all rows with coefficient ± 2 do not occur with non–zero coefficient in any X_l with $i \neq l \neq k$. The claim now follows inductively. ■

Recall that $V(f_i)_i$ denotes the variety defined by $\{f_i = 0\}_i$. A variety is called a *complete intersection* if it can be written as $V(f_1, \ldots, f_d)$ where each inclusion $V(f_1, \ldots, f_k) \subset V(f_1, \ldots, f_{k-1})$ for $k = 2, \ldots, d$ is with codimension one.

PROPOSITION 4.10. *Let M be a cusped orientable 3–manifold which admits a complete hyperbolic structure of finite volume, and let $\mathcal{T}$ be an ideal triangulation* *realising the complete structure. Then $\mathfrak{D}(M)$ is a complete intersection variety of complex dimension equal to the number of cusps of M.*

PROOF. We use the general set–up of Sections 4.1 and 4.2, and let n be the number of tetrahedra, and h be the number of vertices of $\mathcal{T}$.

The deformation variety is defined by the parameter relations and the hyperbolic gluing equations. For each i, the map $(z_i, z'_i, z''_i) \to z_i$ is a bijection between the variety in $(\mathbb{C} - \{0\})^3$ defined by the ideal $I(p_i, p'_i, p''_i)$ and $\mathbb{C} - \{0, 1\}$. Moreover $I(p_i, p'_i, p''_i) = I(p_i, p'_i)$, since:

$$z'_i p_i + (1 - z''_i)p'_i = p''_i.$$

An alternative set of defining relations for the deformation variety is therefore given by $\{p_i, p'_i\}_{i=1,\ldots,n}$ along with the set of hyperbolic gluing equations as given in [**37**], i.e. the equations of the form

$$1 = \prod_{i=1}^{n} (-1)^{a''_{ij}} z_i^{a_{ij} - a''_{ij}} (1 - z_i)^{a''_{ij} - a'_{ij}} =: r_j.$$

The variety defined by the equations $\{p_i = 0, p'_i = 0\}_{i=1,\ldots,n}$ has dimension n, and is clearly a complete intersection.

Recall the definition and properties of the matrix R in [**37**] from the proof of Lemma 4.9. We have $r(R) = n - h$, and the second part of the proof of the lemma applies to R by replacing B_j by R_j, where R_j denotes the row corresponding to the equation $r_j = 1$. We may therefore renumber the rows of R such that $\{R_1, \ldots, R_{n-h}\}$ is a basis for the row space with the property that any other row is an interger linear combination of rows in $\{R_1, \ldots, R_{n-h}\}$, and we renumber the monomials r_j accordingly.

Assume $R_l = \sum_{i=1}^{n-h} \alpha_i R_i$ for $l > n-h$. Then $1 = \prod_{i=1}^{n-h} r_j^{\alpha_i} = \pm r_l$. If $\prod_{i=1}^{n-h} r_j^{\alpha_i} = -r_l$, this implies that $r_l = -1$, which is a contradiction. Hence $\prod_{i=1}^{n-h} r_j^{\alpha_i} = r_l$, and in particular $r_l = 1$ is a consequence of $\{r_1 = 1, \ldots, r_{n-h} = 1\}$. Thus, $\mathfrak{D}(M)$ is defined by

$$\{p_1 = p_1' = \ldots = p_n = p_n' = r_1 - 1 = \ldots = r_{n-h} - 1 = 0\}. \tag{4.18}$$

It follows that $\dim \mathfrak{D}(M) \geq h$ with equality if and only if $\mathfrak{D}(M)$ is a complete intersection. The argument in the proof of Proposition 2.3 of [**37**] applies now, showing that Mostow rigidity implies that $\dim \mathfrak{D}(M) = h$. This completes the proof. ■

COROLLARY 4.11. *Let M be a complete orientable hyperbolic 3–manifold of finite volume with h cusps, and let $\mathcal{T}$ be an ideal triangulation of M with n tetrahedra realising the complete structure. Then the projective solution space $P(M)$ of normal surface Q-theory is a polytope of real dimension $2n + h - 1$.*

PROOF. We only need to verify the dimension. It follows from Proposition 4.10 that $\mathfrak{D}(M)$ has complex dimension h. Hence $\mathfrak{D}_\infty(M) \neq \emptyset$, which using Proposition 4.5 implies that $P(M)$ is non–empty.

We know from Lemma 4.9 that B has rank $n - h$, and hence the set of all solutions to the Q-matching equations is a vector space of real dimension $2n + h$. The intersection of this space with the positive unit simplex is non–empty, and hence of real dimension $2n + h - 1$. ■

4.5. The figure eight knot

Let M denote the complement of the figure eight knot, and M^τ its sister manifold (see Section 2.4). An ideal triangulation for M is shown in Figure 4.17. A triangulation for M^τ can be obtained by reversing the arrows in either of the two tetrahedra. This implies that their deformation varieties are identical.

For both M and M^τ, we have the following hyperbolic gluing equations:

$$1 = (w')^2 w'' (z')^2 z'',$$

$$1 = w^2 w'' z^2 z''.$$

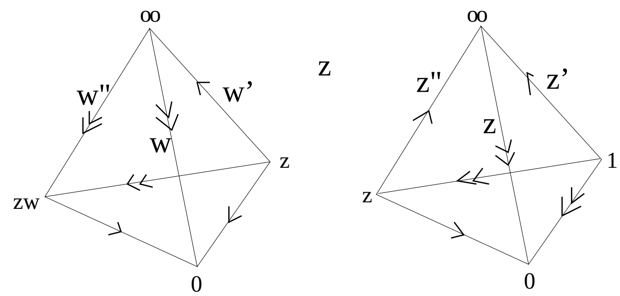


FIGURE 4.17. The triangulation of $m004$

This gives the matrix

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 & 0 & 1 \end{pmatrix}.$$

Hence

$$AC_2 = \begin{pmatrix} -1 & -1 & 2 & -1 & -1 & 2 \\ 1 & 1 & -2 & 1 & 1 & -2 \end{pmatrix} = B,$$

which determines a single Q–matching equation:

$$0 = q + q' - 2q'' + p + p' - 2p''.$$

This agrees with Kang's analysis in [**31**]. One can also work out the induced triangulations of the cusps and determine standard generators for the peripheral subgroups. One obtains: $\nu(\mathcal{L}) = \nu(\mathcal{L}^\tau) = -2q - 2q' + 4q''$, $\nu(\mathcal{M}) = -p' + p'' - q + q''$ and $\nu(\mathcal{M}^\tau) = -p' + p'' + q' - q''$. Note that $\nu(\mathcal{M}^\tau) = \nu(\mathcal{M}) - \frac{1}{2}\nu(\mathcal{L})$. The fundamental solutions and their boundary curves are given in Table 4.1. All normal surfaces corresponding to these vertex solutions are once–punctured Klein bottles (see [**31**]). Note that $\mathcal{N}(M)$ is zero–dimensional, and $P(M)$ is four dimensional. Moreover, the vertex solutions are linearly independent.

4.5.1. Face pairings. Let a fundamental domain for M be given by embedding the tetrahedra in $\mathbb{H}^3$ as indicated by the vertex labels in Figure 4.17. The face

vertex	$\nu(\mathcal{M})$	$\nu(\mathcal{M}^\tau)$	$\nu(\mathcal{L})$	$\mathfrak{BC}(M)$	$\mathfrak{BC}(M^\tau)$
(2,0,0,0,0,1)	1	–1	4	(–4,1)	(4,1)
(0,2,0,0,0,1)	–1	–3	4	(4,1)	(4,3)
(0,0,1,2,0,0)	–1	1	–4	(–4,1)	(4,1)
(0,0,1,0,2,0)	1	3	–4	(4,1)	(4,3)

TABLE 4.1. Normal surface in the figure eight knot complement

pairings associated to $Z = (w, w', w'', z, z', z'') \in \mathfrak{D}(M)$ are defined by assignments of the following ordered triples:

$$A_Z : [0, z, zw] \to [\infty, 1, z]$$

$$\mathcal{M}_Z : [\infty, 0, zw] \to [1, 0, z]$$

$$G_Z : [\infty, zw, z] \to [\infty, 0, 1]$$

It follows that $G_Z = A_Z \mathcal{M}_Z A_Z^{-1}$. We have

$$\pi_1(M) =< A, \mathcal{M} \mid A\mathcal{M}A^{-2}\mathcal{M} = \mathcal{M}A\mathcal{M}A^{-1} > . \tag{4.19}$$

Furthermore, $\mathcal{M}$ is a meridian, and the corresponding longitude is

$$\mathcal{L} = \mathcal{M}^{-1}A\mathcal{M}A^{-1}\mathcal{M}^{-1}A^{-1}\mathcal{M}A.$$

From above face pairings and the equation $z(z-1)w(w-1) = 1$, we obtain representations into $PSL_2(\mathbb{C})$ by putting

$$\rho_Z(\mathcal{M}) = \frac{1}{\sqrt{w(1-z)}} \begin{pmatrix} 1 & 0 \\ 1 & w(1-z) \end{pmatrix}$$

$$\rho_Z(A) = \begin{pmatrix} 1 - wz & -(1-w)^{-1} \\ 1 - w & 0 \end{pmatrix}$$

$$\rho_Z(\mathcal{L}) = \begin{pmatrix} w(w-1) & 0 \\ z(1 + w - w^2)(wz - w - z) & z(z-1) \end{pmatrix}$$

This is actually a representaton into $SL_2(\mathbb{C})$, and the lower right entries in $\mathcal{M}$ and $\mathcal{L}$ correspond to square roots of the holonomies given by Thurston in [**44**], where $H'(\mathcal{M}) = w(1-z)$ and $H'(\mathcal{L}) = z^2(z-1)^2$.

With the above matrices, we can find a map from $\mathfrak{D}(M)$ to the $PSL_2(\mathbb{C})$–character variety. We have $(\operatorname{tr}\mathcal{M})^2 = w + 2(1-wz) + z$ and $\operatorname{tr} A = 1 - wz$. Put $(\operatorname{tr}\mathcal{M})^2 = X$ and $\operatorname{tr} A = y$, and the image of $\mathfrak{D}(M)$ in the $PSL_2(\mathbb{C})$–character variety has equation

$$1 - y - y^2 + (y-1)X = 0,$$

giving a sphere in $\mathbb{C}P^2$. This matches our earlier description, and in particular shows that the map from $\mathfrak{D}(M)$ to $\overline{\mathfrak{X}}_0(M)$ is onto. Note that the smooth projective model of $\mathfrak{D}(M)$ is a torus, and that the map $\mathfrak{D}(M) \to \overline{\mathfrak{X}}(M)$ is generically 2–to–1.

4.5.2. Limiting character. Given the symmetries of M and its the triangulation, it will suffice to consider degeneration of the ideal triangulation to one of the four ideal points of $\mathfrak{D}(M)$. We will consider the point whose associated normal surface coordinates are $(0, 2, 0, 0, 0, 1)$.

We wish to study “geometric” degenerations, i.e. degenerations where both tetrahedra stay positively oriented and only in the limit become flat. The deformation variety is birationally equivalent to the variety in $\mathbb{C}^2$ defined by the single equation:

$$z(1-z)w(1-w) = 1.$$

We can solve the above equation in terms of w, giving

$$z = \frac{1}{2}\Big(1 \pm \sqrt{1 + \frac{4}{w(w-1)}}\Big) \tag{4.20}$$

Since we start deforming from the complete structure where $w_0 = z_0 = \frac{1}{2}(1+\sqrt{-3})$, we take the solution for z with positive sign in front of the root. The desired ideal point corresponds to the degeneration $w \to 0$. Note that then $z \to \infty$ at half the rate. Let $w = w(r) = r^2 w_0$, and obtain a power series expansion of z for r

around zero using equation (4.20). This is of the form $z(r) = \frac{w_0}{r} + \frac{1}{2} + \varphi(r)$, where $\varphi(0) = 0$.

All points on the path $[1, 0) \to \mathfrak{D}(M)$ given by $r \to (w(r), z(r))$ correspond to geometric solutions to the hyperbolic gluing equations (see [**44**]). We can use the face pairings to determine the limiting representations. As $r \to 0$, we have the following limiting traces:

$$\operatorname{tr} \rho_Z(A) \to 1 \qquad \text{and} \qquad \operatorname{tr}^2 \rho_Z(\mathcal{M}) \to \infty,$$

whilst the eigenvalue of $\mathcal{M}^4\mathcal{L}$, which is equal to $-w^2z(1-z)^3$, approaches one. Thus, $\mathcal{M}^4\mathcal{L}$ is an associated boundary slope. Moreover, the image of $\mathcal{M}^4\mathcal{L}$ approaches the identity matrix.

The limiting splitting of M corresponds to a splitting of $M(4, 1)$, which is a graph manifold, along an essential torus (see [**12**]). The limiting pieces admit Seifert fibered structures, and are a twisted I–bundle over the Klein bottle, S_1, and a trefoil knot complement, S_2.

To work out the fundamental groups of the complementary pieces, note that an isomorphism preserving peripheral systems between presentations (2.4) and (4.19) is given by $t \to \mathcal{M}$ and $a \to A$.

Let K' be the punctured Klein bottle in M as pictured in Figure 4.18, and let K be the corresponding Klein bottle in $M(4, 1)$. Identify $I\tilde{\times}K$ with a regular neighbourhood of K in $M(4, 1)$. Standard generators for $\operatorname{im}(\pi_1(K) \to \pi_1(M(4, 1)))$ are $k_1 = \mathcal{M}A^{-2}\mathcal{M}$ and $k_2 = A^{-1}\mathcal{M}A^{-1}$. We have

$$k_2k_1k_2^{-1}k_1 = A\mathcal{M}^{-1}A^{-1}(\mathcal{M}^4\mathcal{L})A\mathcal{M}A^{-1} = 1.$$

Standard generators for the boundary torus of $I\tilde{\times}K$ are $\mathcal{M}_1 = k_1$ and $\mathcal{L}_1 = k_2^2$.

Generators for the complement of $I\tilde{\times}K$ in $M(4, 1)$ are given by $u = A$ and $v = A\mathcal{M}A\mathcal{M}A^{-1}$. We have

$$vu^{-3}v = A\mathcal{M}(\mathcal{M}^4\mathcal{L})\mathcal{M}^{-1}A^{-1} = 1.$$

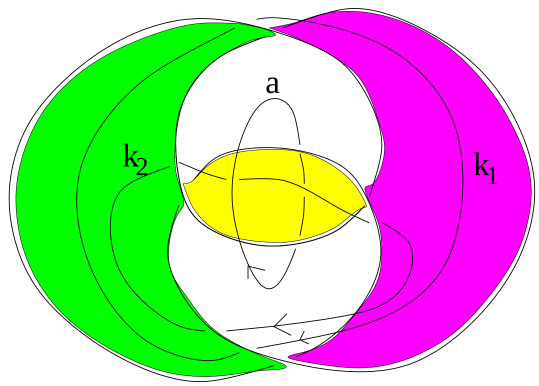


FIGURE 4.18. The punctured Klein bottle

Generators for the boundary torus of the trefoil knot complement are $\mathcal{M}_2 = u^{-1}v$ and $\mathcal{L}_2 = u^3$ (the meridian is standard, but the longitude is not).

The decomposition of the fundamental group of $M(4,1)$ can be worked out from the above. It is an amalgamated product of $\pi_1(S_1) =< k_1, k_2 \mid k_2k_1k_2^{-1}k_1 = 1 >$ and $\pi_1(S_2) =< u, v \mid u^3 = v^2 >$, amalgamated by $\mathcal{M}_1 = \mathcal{M}_2^{-1}$ and $\mathcal{L}_1 = \mathcal{L}_2^{-1}\mathcal{M}_2$.

The limiting representation on the trefoil knot complement is determined by:

$$\rho(A) \to \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \rho(A\mathcal{M}A\mathcal{M}A^{-1}) \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \rho(\mathcal{M}A\mathcal{M}A^{-1}) \to \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

This representation corresponds to a 2–dimensional hyperbolic structure on the base orbifold of S_2, which is a $(2,3,\infty)$–turnover. The limiting representation of the boundary torus is:

$$\rho(\mathcal{M}_2) \to \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \rho(\mathcal{L}_2) \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to obtain a (finite) limiting representation of S_1, we have to conjugate the face pairings by a diagonal matrix with eigenvalues $(rw_0^{-1})^{\pm 1/4}$. The limiting representation is then:

$$\rho(k_1) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho(k_2) \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

giving a cyclic group of order two in $PSL_2(\mathbb{C})$. The limiting image of $\rho(\mathcal{M}_1)$ is E, and the limiting image of $\rho(\mathcal{L}_1)$ is $-E$.

4.6. Remarks

We conclude with present and future directions of research related to the contents of this chapter.

4.6.1. Splitting surfaces. Let ξ be an ideal point of the deformation variety. The example of the preceding section suggests that either the normal surface $S(\xi)$ or (in case that it is not orientable) the boundary of a regular neighbourhood of $S(\xi)$ may be a surface associated to a non–trivial splitting of M. We have already given an interpretation of this phenomenon in the introduction. We said that the tetrahedra associated to the degeneration become very long and thin, and the manifold will split apart in the limit.

With the given parametrisation, quadrilaterals are placed in a tetrahedron σ such that they separate the edges with invariants z converging to one at ξ, and the number of quadrilaterals placed in σ corresponds to the relative growth rate of z', z'' compared to the parameters of the other tetrahedra in $\mathcal{T}$. Geometrically, one can show that the distance in σ between the edges with invariants converging to one is approximately $\log |z'|$ as z' is large, and hence becomes infinite.

We believe that a combinatorial description of the dual tree of $S(\xi)$, as well as a geometric analysis of the limiting representation into $PSL_2(\mathbb{C})$ suffices to show that if the limiting representation corresponds to an ideal point ξ' of the character variety, then the normal surface $S(\xi)$ or (in case that it is not orientable) the boundary of a regular neighbourhood of $S(\xi)$ is indeed a surface associated to

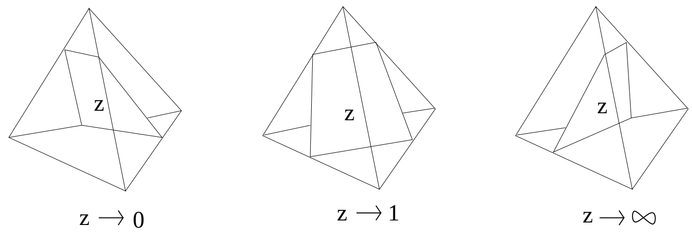


FIGURE 4.19. Quadrilateral types assigned to the different degenerations of an ideal tetrahedron

ξ' according to Culler–Shalen theory. (See [**51**] for a related construction in the 1–cusped case.)

If a degeneration of the ideal triangulation through positively oriented tetrahedra occurs, it would be interesting to see whether one can recover Euclidean (orbifold or cone-manifold) structures on the splitting surface from the geometry of the degeneration, and to show that the normal surface itself is already essential.

4.6.2. Logarithmic limit set. To facilitate the study of large classes of examples, one needs to find an algorithm to compute the logarithmic limit set of the deformation variety, or more generally, a complete intersection variety. This would readily furnish the set of embedded normal surfaces associated to degenerations of ideal triangulations, and determine the set of boundary slopes detected by the deformation variety.

4.6.3. Software. The study of examples is aided by the following software. The program `regina`[**10**] computes normal surfaces, and degenerations of hyperbolic structures can be studied using `SnapPea`[**50**] and `snap/tube/isometric`[**24**].

CHAPTER 5

The Whitehead link

We consider the standard triangulation of the Whitehead link complement $\mathcal{W}$. In Section 5.2 the projective admissible solution space $\mathcal{N}(\mathcal{W})$ is computed, and some normal surfaces are described explicitly. In Section 5.3 the logarithmic limit sets of the deformation variety $\mathfrak{D}(\mathcal{W})$ and the Dehn surgery component $\overline{\mathfrak{E}}_0(\mathcal{W})$ of the $PSL_2(\mathbb{C})$–eigenvalue variety are determined. The varieties and maps needed for this program are contained in Section 5.1, and Section 5.4 contains the results of some computations.

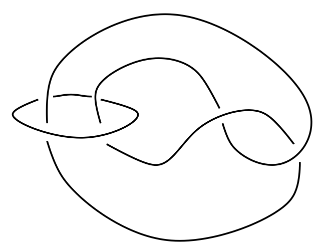


FIGURE 5.1. The Whitehead link

5.1. Associated varieties

The main results of this section are birational equivalences between $\mathfrak{D}(\mathcal{W})$, $\overline{\mathfrak{E}}_0(\mathcal{W})$ and a variety $\mathfrak{C}(\mathcal{W})$ parametrising developing maps.

5.1.1. A note on the manifold. A projection of the right–handed Whitehead link is shown in Figure 5.1, and $\mathcal{W} = \mathcal{W}^+$ denotes its complement in S^3. $\mathcal{W}$ is isometric to the manifold $m129$ in `SnapPea`'s census with the *same* orientation. The complement of a left–handed Whitehead link is oppositely oriented, and will be denoted by $\mathcal{W}^-$.

5.1.2. Triangulation. An ideal triangulation of $\mathcal{W}^-$ is contained in [**44, 36**]. The abstract ideal triangulation used in this thesis is derived from `snap`'s manifold data. The parameter space and the holonomies in [**36**] differ from the ones below since the convention regarding shape parameters is different and the associated link in [**36**] is left-handed.

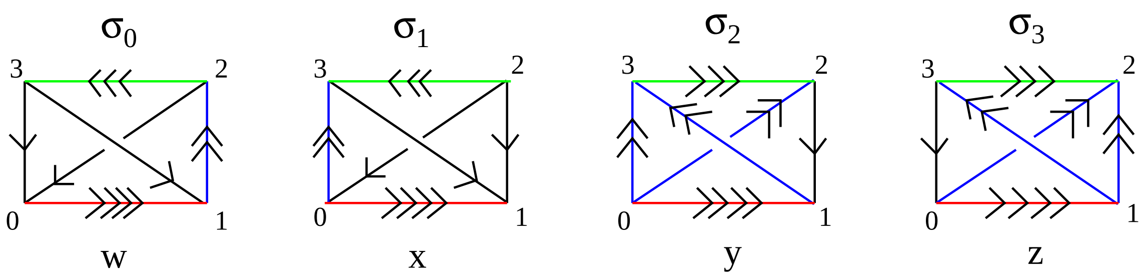


FIGURE 5.2. Triangulation of $\mathcal{W}$

The ideal triangulation of $\mathcal{W}$ is shown in Figure 5.2. There are four tetrahedra, and hence four edges, which will be called red, green, black and blue.[1] The black edge has one arrow, the blue two, the green three and the red four. One of the cusps corresponds to the ideal endpoints of the red edge, and the other to the ideal endpoints of the green edge. The cusps are therefore referred to as the red cusp (also: cusp 0) and the green cusp (also: cusp 1) respectively. The face pairings are decribed in the following table, where the face of tetrahedron σ_i opposite vertex j is denoted by $F_{i,j}$.

$F_{0,0} \to F_{2,0}$	$F_{1,0} \to F_{0,1}$	$F_{2,0} \to F_{0,0}$	$F_{3,0} \to F_{2,1}$
$F_{0,1} \to F_{1,0}$	$F_{1,1} \to F_{3,1}$	$F_{2,1} \to F_{3,0}$	$F_{3,1} \to F_{1,1}$
$F_{0,2} \to F_{1,3}$	$F_{1,2} \to F_{2,3}$	$F_{2,2} \to F_{3,3}$	$F_{3,2} \to F_{0,3}$
$F_{0,3} \to F_{3,2}$	$F_{1,3} \to F_{0,2}$	$F_{2,3} \to F_{1,2}$	$F_{3,3} \to F_{2,2}$

TABLE 5.1. Face pairings

[1]The colours are visible on ps and pdf files.

5.1.3. Deformation variety. The shape parameters are assigned in Figure 5.2, and the following hyperbolic gluing equations can be read off from this figure:

(red & green) $$1 = wxyz$$

(black) $$1 = w'x'y'z'(w'')^2(x'')^2$$

(blue) $$1 = w'x'y'z'(y'')^2(z'')^2.$$

The deformation variety $\mathfrak{D}(\mathcal{W})$ is defined by these equations and the parameter relations:

$$\begin{array}{llll} 1 = w(1-w''), & 1 = x(1-x''), & 1 = y(1-y''), & 1 = z(1-z''), \\ 1 = w'(1-w), & 1 = x'(1-x), & 1 = y'(1-y), & 1 = z'(1-z), \\ 1 = w''(1-w'), & 1 = x''(1-x'), & 1 = y''(1-y'), & 1 = z''(1-z'). \end{array}$$

Note that $\mathfrak{D}(\mathcal{W})$ can be defined by the parameter relations and the equations:

$$1 = wxyz \quad \text{and} \quad w''x'' = y''z''. \tag{5.1}$$

We map $\mathfrak{D}(\mathcal{W})$ into $(\mathbb{C}-\{0\})^4$ by $\varphi(w, w', \ldots, z'') = (w, x, y, z)$. The closure of the resulting image is a variety defined by:

$$0 = 1 - wxyz, \tag{5.2}$$

$$0 = wx(1-y)(1-z) - (1-w)(1-x)yz. \tag{5.3}$$

This variety is the *parameter space* of [**44, 37**], and we denote it by $\mathfrak{D}'(\mathcal{W})$. For any $w, x, y, z \in \mathbb{C} - \{0, 1\}$ subject to (5.2) and (5.3), there is a unique point on $\mathfrak{D}(\mathcal{W})$. Thus, $\mathfrak{D}(\mathcal{W})$ and $\mathfrak{D}'(\mathcal{W})$ are birationally equivalent, and the inverse map:

$$\varphi^{-1}(w,x,y,z) = \left(w, \frac{1}{1-w}, \frac{w-1}{w}, x, \frac{1}{1-x}, \frac{x-1}{x}, y, \frac{1}{1-y}, \frac{y-1}{y}, z, \frac{1}{1-z}, \frac{z-1}{z}\right)$$

is not regular at the intersection of $\mathfrak{D}'(\mathcal{W})$ with the collection of hyperplanes

$$\{w = 1\} \cup \{x = 1\} \cup \{y = 1\} \cup \{z = 1\}$$

in $(\mathbb{C}-\{0\})^4$. If $\mathfrak{D}'(\mathcal{W})$ intersects any one of these hyperplanes, then it intersects either exactly two or four of them.

We can eliminate one of the variables, say w, from the system of equations (5.2, 5.3), and hence there is a map $\varphi' : \mathfrak{D}'(\mathcal{W}) \to (\mathbb{C}-\{0\})^3$ with the closure of its image defined by a single irreducible equation:

$$0 = g(x,y,z) = x - xy - xz + yz - xy^2z^2 + x^2y^2z^2. \tag{5.4}$$

Again, φ' is a birational isomorphism onto its image, and this in particular shows that $\mathfrak{D}(\mathcal{W})$ is irreducible.

5.1.4. Symmetries. There are symmetries in the defining equations of $\mathfrak{D}(\mathcal{W})$, which descend to symmetries in (5.2) and (5.3), and which will be used to abbreviate proofs in the sequel. Consider the following involutions:

$$\tau_1(w,x,y,z) = (z,y,x,w) \qquad \text{and} \qquad \tau_2(w,x,y,z) = (y,z,w,x) \tag{5.5}$$

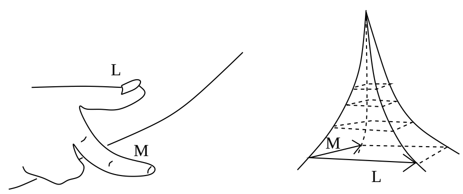
$$\tau_3(w,x,y,z) = (x,w,z,y) \tag{5.6}$$

$$\tau_4(w,x,y,z) = (w,x,z,y) \qquad \text{and} \qquad \tau_5(w,x,y,z) = (x,w,y,z). \tag{5.7}$$

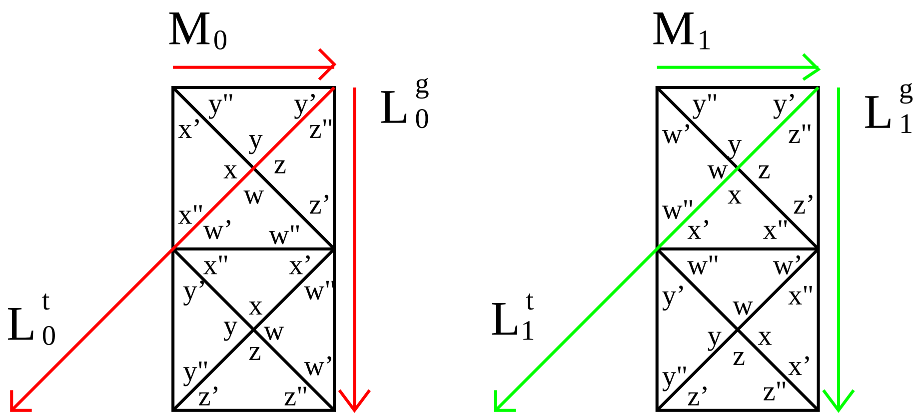
We have $\tau_1\tau_2 = \tau_3 = \tau_4\tau_5$, and each of the pairs (5.5) and (5.7) generates a Kleinian four group. The elements $\tau_6 = \tau_1\tau_4$ and $\tau_7 = \tau_2\tau_4$ have order four, and one can show that the group generated by all involutions is a dihedral group D_4. For any $p \in \mathfrak{D}'(\mathcal{W})$ and any $\tau \in D_4$, $\tau p \in \mathfrak{D}'(\mathcal{W})$, and if $\varphi^{-1} : \mathfrak{D}'(\mathcal{W}) \to \mathfrak{D}(\mathcal{W})$ is regular at p, then it is regular at τp.

5.1.5. A note on orientations. In Figure 5.3(a), the convention for orientations of the peripheral elements is given, where the right–hand rule is applied to a link projection and the induced triangulation of the cusp cross–section is viewed from outside the manifold.

5.1.6. Holonomies. The induced triangulations of the cusp cross–sections as viewed from outside $\mathcal{W}$ are given in Figure 5.3(b). There is a standard meridian,



(a) Orientations of $\mathcal{L}$ and $\mathcal{M}$



(b) Triangulation of cusp cross–section

FIGURE 5.3. Peripheral elements

and we consider two choices for longitudes, which are termed *geometric* and *topological* and denoted by $\mathcal{L}^g$ and $\mathcal{L}^t$ respectively. The geometric longitudes are chosen by `snap` for $m129$. They have the following property. Let $\mathcal{W}(p,q)$ be the manifold obtained from hyperbolic Dehn filling on one of the cusps. The coefficients $(p,q) \in \mathbb{R}^2$ are called *exceptional* if $\mathcal{W}(p,q)$ does not admit a complete hyperbolic metric. If the longitude is geometric, then the set of exceptional coefficients is contained in the rectangle with vertices $\pm(2,\pm 1)$ (see [**30**]). If a null–homologous longitude is chosen, the set of exceptional coefficients is contained in a parallelogram with vertices $\pm(-4,1)$, $\pm(0,1)$ for $\mathcal{W}^-$ (see [**36**]), and $\pm(4,1)$, $\pm(0,1)$ for $\mathcal{W}^+$. In Table 5.2, we show the Dehn surgery coefficients with respect to the indicated peripheral systems which yield homeomorphic manifolds.

$\{\mathcal{M}, \mathcal{L}^t\}$ for $\mathcal{W}^-$	$\{\mathcal{M}, \mathcal{L}^g\}$ for $\mathcal{W}^+$ and $\mathcal{W}^-$	$\{\mathcal{M}, \mathcal{L}^t\}$ for $\mathcal{W}^+$
(p+2q, -q)	(p,q)	(p+2q,q)

TABLE 5.2. Surgery coefficients which yield homeomorphic manifolds

The derivatives of holonomies can be read off from Figure 5.3, and simplify to:

$$H'(\mathcal{M}_0) = \frac{z}{w'y''} = \frac{x'z''}{y}, \quad H'(\mathcal{L}_0^t) = x^2y^2, \quad H'(\mathcal{L}_0^g) = \frac{(w'')^2}{(y'')^2} = \frac{(z'')^2}{(x'')^2}, \tag{5.8}$$

$$H'(\mathcal{M}_1) = \frac{w'z''}{y} = \frac{x}{w''z'}, \quad H'(\mathcal{L}_1^t) = w^2y^2, \quad H'(\mathcal{L}_1^g) = \frac{(z'')^2}{(w'')^2} = \frac{(x'')^2}{(y'')^2}. \tag{5.9}$$

The holonomies yield that the complete structure is attained at $(w, x, y, z) = (i, i, i, i)$. The action of D_4 on the holonomies is described in Table 5.3, where $(m_i, l_i) = (H'(\mathcal{M}_i), H'(\mathcal{L}_i^t))$. The relationship between elements of D_4 and elements of the symmetry group of $\mathcal{W}$ can be deduced from this table.

	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7
(m_0, l_0)	(m_0, l_0)	(m_0^{-1}, l_0^{-1})	(m_0^{-1}, l_0^{-1})	(m_1^{-1}, l_1^{-1})	(m_1, l_1)	(m_1, l_1)	(m_1^{-1}, l_1^{-1})
(m_1, l_1)	(m_1^{-1}, l_1^{-1})	(m_1, l_1)	(m_1^{-1}, l_1^{-1})	(m_0^{-1}, l_0^{-1})	(m_0, l_0)	(m_0^{-1}, l_0^{-1})	(m_0, l_0)

TABLE 5.3. Action of D_4 on holonomies

5.1.7. Developing map. Given a point $Z = (w, w', \ldots, z'') \in \mathfrak{D}(\mathcal{W})$, choose isometric embeddings of the four tetrahedra into $\mathbb{H}^3$ such that they join up around the red edge, which we embed as the geodesic $[\infty, 0]$. Precisely, we start with an embedding of σ_0 with vertices at the points $0, 1, \infty, w$ on the sphere at infinity, and then embed the other tetrahedra according to the parameters given by Z using the face pairings in Table 5.4.

$F_{0,2} \to F_{1,3}$	$F_{1,2} \to F_{2,3}$	$F_{2,2} \to F_{3,3}$	$F_{3,2} \to F_{0,3}$
$F_{0,3} \to F_{3,2}$	$F_{1,3} \to F_{0,2}$	$F_{2,3} \to F_{1,2}$	$F_{3,3} \to F_{2,2}$

TABLE 5.4. Face pairings used for fundamental domain

The resulting ideal vertices of the tetrahedra are indicated in Figure 5.4. Note that some tetrahedra may intersect or be "inverted". If all tetrahedra are positively oriented, then the resulting fundamental domain is an ideal octahedron which is shown in Figure 5.5.

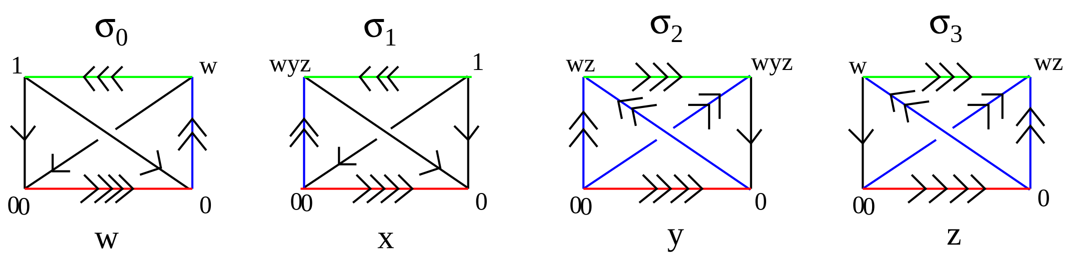
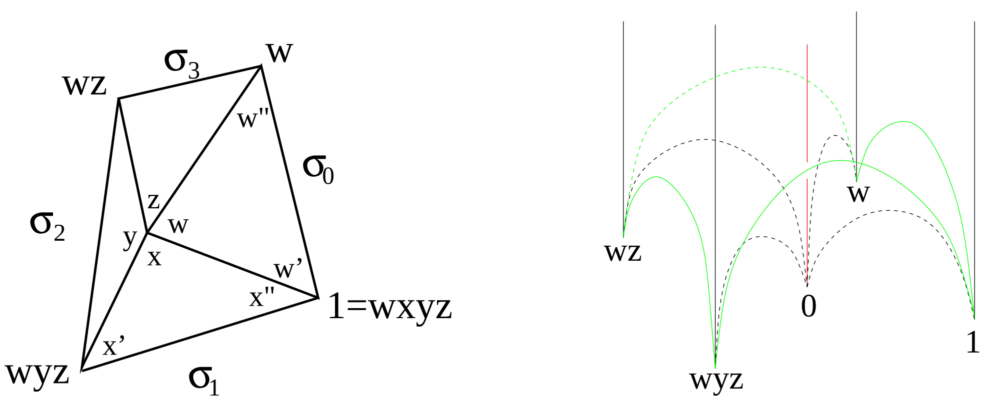


FIGURE 5.4. Developing map: general case



(a) Horospherical cross section

(b) Fundamental domain

FIGURE 5.5. Developing map: positively oriented tetrahedra

The remaining face pairings:

$$\mathfrak{a}_Z : [0, 1, wyz] \to [\infty, w, 1], \qquad \mathfrak{b}_Z : [w, \infty, wz] \to [1, \infty, wyz],$$

$$\mathfrak{c}_Z : [0, 1, w] \to [0, wyz, wz], \qquad \mathfrak{c}_Z\mathfrak{d}_Z : [w, 0, wz] \to [wz, \infty, wyz],$$

determine a representation of $\pi_1(\mathcal{W})$ into $PSL_2(\mathbb{C})$:

$$\mathfrak{a}_Z = \sqrt{\frac{w'}{x'}} \begin{pmatrix} wx''+x' & -\frac{x'}{w'} \\ 1 & 0 \end{pmatrix} \qquad \mathfrak{b}_Z = \sqrt{\frac{x'z''}{y}} \begin{pmatrix} \frac{y}{x'z''} & 1-x''z' \\ 0 & 1 \end{pmatrix}$$

$$\mathfrak{c}_Z = \sqrt{\frac{w''y'}{x}} \begin{pmatrix} 1 & 0 \\ xw'(1-wy) & \frac{x}{w''y'} \end{pmatrix} \qquad \mathfrak{d}_Z = \sqrt{\frac{y}{w'z''}} \begin{pmatrix} (1-wx)w' & \frac{y''}{w''} \\ \frac{w'}{x'} & \frac{1-xz}{w''} \end{pmatrix}$$

At the complete structure, we have:

$$\mathfrak{a}_0 = \begin{pmatrix} 1+i & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathfrak{b}_0 = \begin{pmatrix} 1 & 1-i \\ 0 & 1 \end{pmatrix}$$

$$\mathfrak{c}_0 = \begin{pmatrix} 1 & 0 \\ -1+i & 1 \end{pmatrix} \qquad \mathfrak{d}_0 = \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix}$$

5.1.8. Fundamental group. `snap` computes an abstract presentation of $\pi_1(\mathcal{W})$:

$$\pi_1(\mathcal{W}) =< \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \mid \mathfrak{d} = \mathfrak{a}\mathfrak{b}, \mathfrak{b}\mathfrak{c}\mathfrak{d} = \mathfrak{c}\mathfrak{d}\mathfrak{c}, \mathfrak{a}\mathfrak{c} = \mathfrak{b}\mathfrak{a} >,$$

and the peripheral elements:

$$\mathcal{M}_0 = \mathfrak{b}^{-1}, \qquad \mathcal{L}_0^g = \mathfrak{a}\mathfrak{d}^{-1}\mathfrak{c}^{-1}, \qquad \mathcal{L}_0^t = \mathcal{M}_r^{-2}\mathcal{L}_r^g,$$

$$\mathcal{M}_1 = \mathfrak{d}, \qquad \mathcal{L}_1^g = \mathfrak{a}^2\mathfrak{b}\mathfrak{c}, \qquad \mathcal{L}_1^t = \mathcal{M}_g^{-2}\mathcal{L}_g^g.$$

$\pi_1(\mathcal{W})$ can be generated by the two meridians, and one obtains a single relation:

$$\begin{aligned} \pi_1(\mathcal{W}) =< \mathcal{M}_0, \mathcal{M}_1 \mid & \mathcal{M}_1\mathcal{M}_0\mathcal{M}_1\mathcal{M}_0^{-1}\mathcal{M}_1^{-1}\mathcal{M}_0^{-1}\mathcal{M}_1\mathcal{M}_0 \\ & = \mathcal{M}_0\mathcal{M}_1\mathcal{M}_0^{-1}\mathcal{M}_1^{-1}\mathcal{M}_0^{-1}\mathcal{M}_1\mathcal{M}_0\mathcal{M}_1 > \end{aligned} \tag{5.10}$$

For each $Z \in \mathfrak{D}(\mathcal{W})$, the assignment of face pairings to generators defines a representation of $\pi_1(\mathcal{W})$ into $PSL_2(\mathbb{C})$ by $\overline{\rho}(\gamma) = \gamma_Z$, and the squares of the upper left entries in the matrices representing the meridians and longitudes are the holonomies in (5.8) and (5.9). We obtain a map $\Theta : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{X}}(\mathcal{W})$ defined by:

$$\begin{aligned}\Theta(w, w', \ldots, z'') =&(\mathrm{tr}^2 \overline{\rho}\mathcal{M}_0, \mathrm{tr}^2 \overline{\rho}\mathcal{M}_1, \mathrm{tr}^2 \overline{\rho}\mathcal{M}_0\mathcal{M}_1) \\ &= \Big(\frac{w''y'}{x} + 2 + \frac{x}{w''y'}, \frac{w''z'}{x} + 2 + \frac{x}{w''z'}, \frac{(yz-1)^2}{(y-1)(z-1)}\Big).\end{aligned}$$

We will show that the map $\Theta : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{X}}(\mathcal{W})$ has degree 4. This corresponds to the observation that $\mathcal{W}(p, q, p', q')$, $\mathcal{W}(-p, -q, p', q')$, $\mathcal{W}(p, q, -p', -q')$, and $\mathcal{W}(-p, -q, -p', -q')$ are geometrically distinguished (they "spiral in different directions into the cusps"), whilst they have isomorphic fundamental groups.

5.1.9. We compute the variety $\mathfrak{C}(\mathcal{W})$ of Subsection 1.1.8 from (5.10):

$$(5.11) \qquad \rho(\mathcal{M}_0) = \begin{pmatrix} s & c \\ 0 & s^{-1}\end{pmatrix} \text{ and } \rho(\mathcal{M}_1) = \begin{pmatrix} u & 0 \\ 1 & u^{-1}\end{pmatrix} \text{subject to}$$

$$\begin{aligned}(5.12) \quad 0 &= f(s, u, c) \\ &= (s - s^{-1})(u - u^{-1}) + c(s^{-2}u^{-2} - u^{-2} - s^{-2} + 4 - s^2 - u^2 + s^2u^2) \\ &\quad + c^2(2s^{-1}u^{-1} - su^{-1} - s^{-1}u + 2su) + c^3 \in \mathbb{C}[s^{\pm 1}, u^{\pm 1}, c].\end{aligned}$$

$\mathfrak{C}(\mathcal{W})$ is an irreducible hypersurface in $(\mathbb{C} - \{0\})^2 \times \mathbb{C}$, and its intersection with $c = 0$ is the collection of lines $\{s^2 = 1, c = 0\} \cup \{u^2 = 1, c = 0\}$, which parametrises reducible representations. Moreover, $\mathfrak{C}(\mathcal{W})$ is a cover of the Dehn surgery component $\mathfrak{X}_0(\mathcal{W})$ of the character variety, since any irreducible representation of $\pi_1(\mathcal{W})$ into $SL_2(\mathbb{C})$ is conjugate to an element of $\mathfrak{C}(\mathcal{W})$, and it is a 4–to–1 branched cover of $\mathfrak{X}_0(\mathcal{W})$ since $\mathfrak{C}(\mathcal{W})$ is not contained in the union of hypersurfaces $s^2 = 1$ and $u^2 = 1$.

If $f(s, u, c) = 0$, then $f(-s, u, -c) = f(s, -u, -c) = f(-s, -u, c) = 0$, and the four solutions correspond to the action of $\mathrm{Hom}(\pi_1(\mathcal{W}), \mathbb{Z}_2)$ on $\mathfrak{C}(\mathcal{W})$. To obtain a description of the corresponding quotient map $\mathfrak{C}(\mathcal{W}) \to \overline{\mathfrak{C}}(\mathcal{W})$, and hence a parametrisation of $\overline{\mathfrak{C}}(\mathcal{W})$, note that (5.11) can be adjusted by a conjugation and rewritten in the form:

$$\rho(\mathcal{M}_0) = \frac{1}{s}\begin{pmatrix} s^2 & cus \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(\mathcal{M}_1) = \frac{1}{u}\begin{pmatrix} u^2 & 0 \\ 1 & 1 \end{pmatrix}, \tag{5.13}$$

which is still subject to $0 = f(s,u,c)$. The map $\mathrm{q}_1 : (\mathbb{C}-\{0\})^2\times\mathbb{C} \to (\mathbb{C}-\{0\})^2\times\mathbb{C}$ defined by $\mathrm{q}_1(s,u,c) = (s^2,u^2,suc)$ can be identified with the natural quotient map $\mathfrak{C}(\mathcal{W}) \to \overline{\mathfrak{C}}(\mathcal{W})$, and the defining equation for $\overline{\mathfrak{C}}(\mathcal{W})$ can be derived from this relationship. $\overline{\mathfrak{C}}(\mathcal{W})$ can be viewed as a variety of representations into $GL_2(\mathbb{C})$:

$$\overline{\rho}_{GL}(\mathcal{M}_0) = \begin{pmatrix} \overline{s} & d \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{\rho}_{GL}(\mathcal{M}_1) = \begin{pmatrix} \overline{u} & 0 \\ 1 & 1 \end{pmatrix} \quad \text{subject to} \tag{5.14}$$

$$\begin{aligned} 0 = {} & \overline{su}(\overline{s}-1)(\overline{u}-1) + d(1-\overline{s}-\overline{u}+4\overline{su}-\overline{s}^2\overline{u}-\overline{su}^2+\overline{s}^2\overline{u}^2) \\ & + d^2(2-\overline{s}-\overline{u}+2\overline{su}) + d^3 \in \mathbb{C}[\overline{s}^{\pm1},\overline{u}^{\pm1},d]. \end{aligned} \tag{5.15}$$

For each $\overline{\rho}_{GL}$ as above, there is a unique $PSL_2(\mathbb{C})$–representation $\overline{\rho}$ such that $\overline{\rho}_{GL}$ and $\overline{\rho}$ are identical as representations into the group of projective transformations of $\mathbb{C}P^1$. Hence there is a 1–1 correspondence between $GL_2(\mathbb{C})$–representations of $\pi_1(\mathcal{W})$ of the form (5.14) and irreducible $PSL_2(\mathbb{C})$–representations of $\pi_1(\mathcal{W})$ which lift to $SL_2(\mathbb{C})$.

5.1.10. $\mathfrak{D}(\mathcal{W})$ has a singularity at infinity at the ideal point where $w,x,y,z \to 1$, and $\mathfrak{D}'(\mathcal{W})$ is singular at the corresponding point $(w,x,y,z) = (1,1,1,1)$. The variety $\overline{\mathfrak{C}}(\mathcal{W})$ is an irreducible variety without singularities, and we now show that it parametrises developing maps when thought of as a variety in $(\mathbb{C}-\{0\})^3$. We show in Subsection 5.4.6 that the intersection of $\overline{\mathfrak{C}}(\mathcal{W})$ with $c=0$ corresponds to the ideal point of $\mathfrak{D}(\mathcal{W})$ where $w,x,y,z\to 1$.

LEMMA 5.1. *There is a birational isomorphism* $\Phi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{C}}(\mathcal{W})$.

PROOF. To construct the map $\Phi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{C}}(\mathcal{W})$, conjugate the face pairings $(\mathfrak{a}_Z - \mathfrak{d}_Z)$ by a suitably chosen matrix A to obtain a form analogous to (5.13), and then adjust the resulting representation by multiplication:

$$\overline{\rho}'_Z(\mathcal{M}_0) := \sqrt{\frac{x}{w''y'}}A\mathfrak{b}_Z^{-1}A^{-1} = \begin{pmatrix} \frac{w''y'}{x} & z-1 \\ 0 & 1 \end{pmatrix}$$

$$\text{and}\quad \overline{\rho}'_Z(\mathcal{M}_1) := \sqrt{\frac{w''z'}{x}}A\mathfrak{d}_Z A^{-1} = \begin{pmatrix} \frac{x}{w''z'} & 0 \\ 1 & 1 \end{pmatrix}.$$

It can be verified that $\overline{\rho}'_Z$ defines a $GL_2(\mathbb{C})$–representation for each $Z \in \mathfrak{D}(\mathcal{W})$. This induces a map $\Phi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{C}}(\mathcal{W})$ defined by:

$$\Phi(w, w', \dots, z'') = \big(\frac{w''y'}{x}, \frac{x}{w''z'}, z-1\big). \tag{5.16}$$

An elementary calculation shows that this map is 1–1 and that its image is dense in $\overline{\mathfrak{C}}(\mathcal{W})$. A rational inverse is given by the following map $\overline{\mathfrak{C}}(\mathcal{W}) \to \mathfrak{D}'(\mathcal{W})$:

$$(\overline{s}, \overline{u}, d) \to \left(\frac{d + d^2 - d\overline{s} + \overline{su} + d\overline{su}}{(1+d)(d+\overline{su})}, \frac{\overline{su}}{d + d^2 - d\overline{s} + \overline{su} + d\overline{su}}, 1 + \frac{d}{\overline{su}}, 1+d\right), \tag{5.17}$$

which composed with $\varphi : \mathfrak{D}'(\mathcal{W}) \to \mathfrak{D}(\mathcal{W})$ gives a map $\Phi^{-1} : \overline{\mathfrak{C}}(\mathcal{W}) \to \mathfrak{D}(\mathcal{W})$. Composition of the maps Φ and Φ^{-1} can readily be observed to induce the identity on both $\mathfrak{D}(\mathcal{W})$ and $\overline{\mathfrak{C}}(\mathcal{W})$, and this proves the lemma. ■

COROLLARY 5.2. $\Theta : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{X}}_0(\mathcal{W})$ *is generically 4–to–1 and onto.*

PROOF. Note that $\Theta : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{X}}(\mathcal{W})$ factors through $\overline{\mathfrak{C}}(\mathcal{W})$, and hence it is enough to show that the map $\overline{\mathfrak{C}}(\mathcal{W}) \to \overline{\mathfrak{X}}_0(\mathcal{W})$ is generically 4–to–1 and onto. Firstly, $\overline{\mathfrak{C}}(\mathcal{W})$ is irreducible, and hence its image in $\overline{\mathfrak{X}}(\mathcal{W})$ is irreducible. Secondly, the above lemma implies that $\overline{\mathfrak{C}}(\mathcal{W})$ contains a discrete and faithful representation, and hence maps to the Dehn surgery component. Since there are $\overline{\rho} \in \overline{\mathfrak{C}}(\mathcal{W})$ such that $\text{tr}^2\,\overline{\rho}(\mathcal{M}_0) \neq 4 \neq \text{tr}^2\,\overline{\rho}(\mathcal{M}_1)$, there are generically four elements of each conjugacy class of representations contained in $\overline{\mathfrak{C}}(\mathcal{W})$ (see Subsection 1.1.8). Thus, the degree of Θ is 4. ■

5.1.11. Eigenvalue maps. Recall that $\mathfrak{E}(\mathcal{W})$ and $\overline{\mathfrak{E}}(\mathcal{W})$ denote the respective eigenvalue varieties. We wish to show that the subvariety $\overline{\mathfrak{E}}_0(\mathcal{W})$ corresponding to $\overline{\mathfrak{X}}_0(\mathcal{W})$ is birationally equivalent to $\mathfrak{D}(\mathcal{W})$. The following lemma establishes suitable affine coordinates and maps needed for this.

LEMMA 5.3. *There are a quotient map* $\mathrm{q}_3 : \mathfrak{E}(\mathcal{W}) \to \overline{\mathfrak{E}}(\mathcal{W})$ *corresponding to the* $\mathrm{Hom}(\pi_1(\mathcal{W}), \mathbb{Z}_2)$ *action, and* eigenvalue maps $\mathrm{e} : \mathfrak{C}(\mathcal{W}) \to \mathfrak{E}(\mathcal{W})$ *and* $\overline{\mathrm{e}} : \overline{\mathfrak{C}}(\mathcal{W}) \to \overline{\mathfrak{E}}(\mathcal{W})$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{C}(\mathcal{W}) & \xrightarrow{\ \mathrm{e}\ } & \mathfrak{E}_0(\mathcal{W}) \\ {\scriptstyle \mathrm{q}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{q}_3} \\ \overline{\mathfrak{C}}(\mathcal{W}) & \xrightarrow{\ \overline{\mathrm{e}}\ } & \overline{\mathfrak{E}}_0(\mathcal{W}) \end{array}$$

PROOF. Let $\vartheta_\gamma : \mathfrak{C}(\mathcal{W}) \to \mathbb{C}$ be the holomorphic map which takes ρ to the upper left entry of $\rho(\gamma)$. Then define $\mathrm{e} : \mathfrak{C}(\mathcal{W}) \to \mathfrak{E}(\mathcal{W})$ to be the map

$$\mathrm{e}(\rho) = (\vartheta_{\mathcal{M}_0}(\rho), \vartheta_{\mathcal{L}_0^t}(\rho), \vartheta_{\mathcal{M}_1}(\rho), \vartheta_{\mathcal{L}_1^t}(\rho)). \tag{5.18}$$

Since every $\rho \in \mathfrak{C}(\mathcal{W})$ is triangular on the peripheral subgroups, it follows that this map is well–defined. Denote the affine coordinates of $\mathfrak{E}(\mathcal{W})$ by (s, t, u, v), so they correspond to eigenvalues of $\mathcal{M}_0$, $\mathcal{L}_0^t$, $\mathcal{M}_1$ and $\mathcal{L}_1^t$.

Denote the map which takes $\overline{\rho}_{GL}$ to the upper left entry of $\overline{\rho}_{GL}(\gamma)$ by ϑ_γ as well, and let $\overline{\mathrm{e}} : \overline{\mathfrak{C}}(\mathcal{W}) \to \overline{\mathfrak{E}}(\mathcal{W})$ be the map

$$\overline{\mathrm{e}}(\overline{\rho}_{GL}) = (\vartheta_{\mathcal{M}_0}(\overline{\rho}_{GL}), \vartheta_{\mathcal{L}_0^t}(\overline{\rho}_{GL}), \vartheta_{\mathcal{M}_1}(\overline{\rho}_{GL}), \vartheta_{\mathcal{L}_1^t}(\overline{\rho}_{GL})) \tag{5.19}$$

We have to check that this map has the right range. Note that the upper left entries of $\overline{\rho}_{GL}(\mathcal{M}_0)$ and $\overline{\rho}_{GL}(\mathcal{M}_1)$ are the squares of the eigenvalues of matrices representing the associated $PSL_2(\mathbb{C})$–representation. The longitudes are the following words in the meridians:

$$\mathcal{L}_0^t = \mathcal{M}_0^{-1}\mathcal{M}_1\mathcal{M}_0\mathcal{M}_1^{-1}\mathcal{M}_0^{-1}\mathcal{M}_1^{-1}\mathcal{M}_0\mathcal{M}_1 \tag{5.20}$$

$$\mathcal{L}_1^t = \mathcal{M}_1^{-1}\mathcal{M}_0\mathcal{M}_1\mathcal{M}_0^{-1}\mathcal{M}_1^{-1}\mathcal{M}_0^{-1}\mathcal{M}_1\mathcal{M}_0, \tag{5.21}$$

and hence $\overline{\rho}_{GL}(\mathcal{L}_i^t) = \overline{\rho}(\mathcal{L}_i^t)$ for any $\overline{\rho}_{GL}$ and its corresponding unique $PSL_2(\mathbb{C})$–representation $\overline{\rho}$. In particular, $\vartheta_{\mathcal{L}_i^t}(\overline{\rho}_{GL})$ is an eigenvalue of $\overline{\rho}_{GL}(\mathcal{L}_i^t) = \overline{\rho}(\mathcal{L}_i^t)$ and it is independent of the choice of the signs of matrices representing $\overline{\rho}(\mathcal{M}_0)$ and $\overline{\rho}(\mathcal{M}_1)$. It follows that we can give $\overline{\mathfrak{E}}(\mathcal{W})$ the affine coordinates $(\overline{s}, t, \overline{u}, v)$.

The natural quotient map which makes the above diagram commute is therefore $\mathrm{q}_3 : \mathfrak{E}(\mathcal{W}) \to \overline{\mathfrak{E}}(\mathcal{W})$ defined by $\mathrm{q}_3(s,t,u,v) = (s^2, t, u^2, v)$. This map clearly corresponds to the $\mathrm{Hom}(\pi_1(\mathcal{W}), \mathbb{Z}_2)$ action on $\mathfrak{E}(\mathcal{W})$.

Since any irreducible $SL_2(\mathbb{C})$–representation is conjugate to a representation in $\mathfrak{C}(\mathcal{W})$, the closure of the image of e is the component of $\mathfrak{E}(\mathcal{W})$ corresponding to the Dehn surgery component $\mathfrak{X}_0(\mathcal{W})$. It follows that the closure of the image of $\overline{\mathrm{e}}$ corresponds to $\overline{\mathfrak{X}}_0(\mathcal{W})$. In particular the composite mapping $\Psi = \overline{\mathrm{e}} \circ \Phi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{E}}_0(\mathcal{W})$ is onto. ■

5.1.12. Holonomy variety. From the face pairings $(\mathfrak{a}_Z - \mathfrak{d}_Z)$, one may compute that $\vartheta_{\mathcal{M}_0}(\overline{\rho}_Z) = \frac{x'z''}{y}$, $\vartheta_{\mathcal{L}_0^t}(\overline{\rho}_Z) = xy$, $\vartheta_{\mathcal{M}_1}(\overline{\rho}_Z) = \frac{w'z''}{y}$ and $\vartheta_{\mathcal{L}_1^t}(\overline{\rho}_Z) = wy$. The map $\Psi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{E}}(\mathcal{W})$ with respect to the chosen coordinates is therefore defined by

$$\Psi(w, w', w'', x, x', x'', y, y', y'', z, z', z'') = \big(\frac{x'z''}{y}, xy, \frac{w'z''}{y}, wy\big) \tag{5.22}$$

Note that at the complete structure $\Psi\varphi^{-1}(i,i,i,i) = (1,-1,1,-1)$.

LEMMA 5.4. *The map $\Psi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{E}}_0(\mathcal{W})$ is a birational isomorphism.*

PROOF. Since Ψ is a regular map and $\mathfrak{D}(\mathcal{W})$ is irreducible, it follows that the image, which we have identified as $\overline{\mathfrak{E}}_0(\mathcal{W})$, is irreducible. Thus, $\Psi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{E}}_0(\mathcal{W})$ is a regular map of irreducible varieties. It remains to show that it has degree one.

Assume that $\Psi\varphi^{-1}(w_0, x_0, y_0, z_0) = \Psi\varphi^{-1}(w_1, x_1, y_1, z_1)$ where (w_i, x_i, y_i, z_i) are two regular points of φ on $\mathfrak{D}'(\mathcal{W})$. An elementary calculation shows that the points are either identical or that we have $w_0 = x_0$ and $w_1 = x_1$. Thus, all points on which Ψ does not have degree one are contained on the hypersurface $w = x$. Since for any

$w \in \mathbb{C} - \{0, \pm 1\}$, the point $\varphi^{-1}(w, -w^{-1}, w, -w^{-1})$ is contained in $\mathfrak{D}(\mathcal{W})$, $\mathfrak{D}(\mathcal{W})$ is not contained in the hypersurface $w = x$, and this completes the proof of the lemma. ■

An inverse $\overline{\mathfrak{E}}_0(\mathcal{W}) \to \mathfrak{D}(\mathcal{W})$ taking $(\overline{s}, t, \overline{u}, v) \to (w, w', \ldots, z'')$ is determined by the following map $\overline{\mathfrak{E}}_0(\mathcal{W}) \to \mathfrak{D}'(\mathcal{W})$ which can be computed from (5.22):

$$(\overline{s}, t, \overline{u}, v) \to \left(\frac{v(\overline{s} - \overline{u})}{\overline{s}t - \overline{u}v}, \frac{t(\overline{s} - \overline{u})}{\overline{s}t - \overline{u}v}, \frac{\overline{s}t - \overline{u}v}{\overline{s} - \overline{u}}, \frac{\overline{s}t - \overline{u}v}{tv(\overline{s} - \overline{u})} \right).$$

This map is not regular on a 1–dimensional subvariety of $\overline{\mathfrak{E}}_0(\mathcal{W})$, which is defined by the following three equations:

$$(5.23) \qquad \overline{s} = \overline{u}, \qquad t = v, \qquad 0 = \overline{s} - t + \overline{s}t + \overline{s}^2 t - \overline{s}^2 t^2 - \overline{s}^3 t^2 + \overline{s}^4 t^2 - \overline{s}^3 t^3.$$

See Subsection 5.4.3 for a computation of this subvariety.

5.2. Embedded normal surfaces

The projective admissible solution space $\mathcal{N}(\mathcal{W})$ is computed in this section and surfaces corresponding to vertex solutions are determined, as well as their boundary curves. The results are compared with the complete description of incompressible surfaces in $\mathcal{W}$ in [**21**], and the boundary curve space $\mathfrak{BC}(\mathcal{W})$ in [**32**].

5.2.1. We use the convention for the quadrilateral coordinates of Chapter 4. Table 5.5 indicates the position of the quadrilaterals in the tetrahedra. The notation ij/kl means that the particular quadrilateral type separates the vertices i and j from the vertices k and l.

Quadrilateral	q_i	q_i'	q_i''
Separates	01/23	03/12	02/13

TABLE 5.5. Quadrilateral types

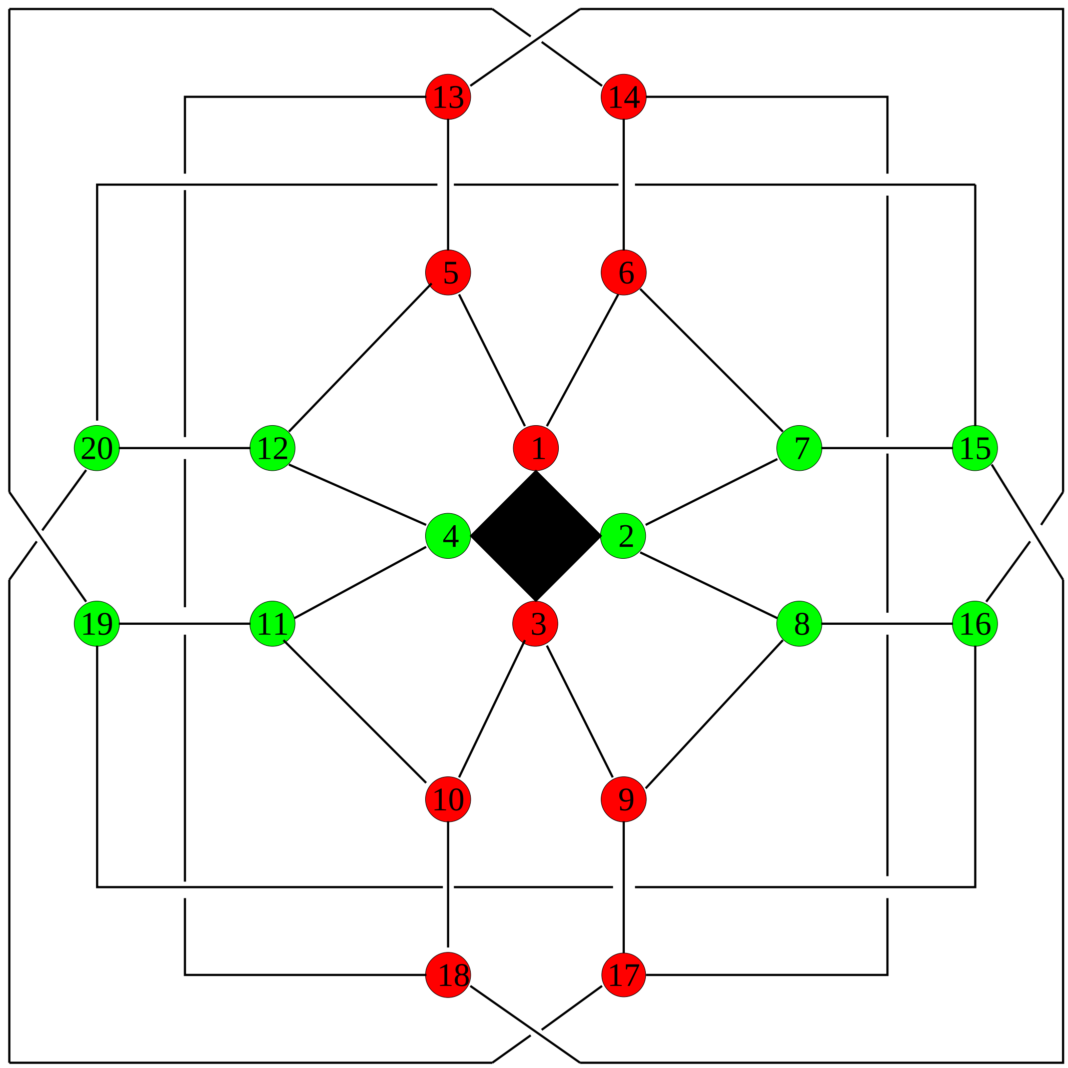


FIGURE 5.6. Projective admissible solution space $\mathcal{N}(\mathcal{W})$

The Q–matching equations can be worked out directly from the triangulation or using the matrix relation from Chapter 4. There are the following two equations:

$$0 = q_0' - q_0'' + q_1' - q_1'' + q_2' - q_2'' + q_3' - q_3'' \tag{5.24}$$

$$0 = -q_0 + q_0' - q_1 + q_1' + q_2 - q_2' + q_3 - q_3' \tag{5.25}$$

The vertices of $\mathcal{N}(\mathcal{W})$ can be obtained using `SnapPea`'s triangulation of $m129$ and `regina`. However, we are interested in a complete description of $\mathcal{N}(\mathcal{W})$. An embedded normal surface intersects each tetrahedron in quadrilaterals of at most one quadrilateral type. In order to find projective admissible solutions, one therefore sets all variables but one from each tetrahedron equal to zero, and computes solutions for the resulting equations in the positive unit simplex. Since the chosen triangulation of $\mathcal{W}$ has four tetrahedra, one obtains 81 systems of linear equations. We give an example.

From the above system (5.24, 5.25), one gets a single equation in the variables q_0, q_1, q_2 and q_3. Together with the normalisation condition, one has:

$$\begin{aligned} 0 &= -q_0 - q_1 + q_2 + q_3, \\ 1 &= q_0 + q_1 + q_2 + q_3, \\ \text{subject to} \qquad & q_0, q_1, q_2, q_3 \geq 0. \end{aligned}$$

This in fact gives a square of solutions:

$$\big(-q_1 + \frac{1}{2}, 0, 0, q_1, 0, 0, \frac{1}{2} - q_3, 0, 0, q_3, 0, 0\big) \text{ where } q_1, q_3 \in [0, \frac{1}{2}].$$

All other solutions consist either of a segment or the empty set. They can be pieced together, and the resulting set, which is $\mathcal{N}(\mathcal{W})$, is shown in Figure 5.6.

There are twenty vertex solutions, which we denote by $V_1, \ldots, V_{20}$. We rescale each vertex solution V_i to obtain a (minimal) integer solution, and denote the resulting normal surface by F_i. These normal surfaces are described in Table 5.6. The information given in the table is explained in Subsection 5.2.3.

5.2.2. Identifying surfaces. Using the explicit description of normal surfaces in terms of Q–discs and T–discs, one can work out the topological type and the position of normal surfaces in the manifold. The pictures of the gluing pattern of some surfaces are shown in Figure 5.7, where we only use the quadrilaterals and finitely many triangles to obtain compact surfaces, along whose boundary components infinite normal annuli have to be attached. The boundary components are drawn in the colour of the corresponding cusp. We introduce triangle coordinates by numbering the vertices of the four tetrahedra from 0 to 15.

5.2.3. Embedded normal surfaces. We now describe the contents of Table 5.6. First, the normal Q–coordinate $N(F_i)$ is given, then the topological type of F_i, where T_1 stands for a once–punctured torus, S_3 for a thrice–punctured sphere, R_2 for a twice–punctured $\mathbb{R}P^2$. The abbreviations K_i and T_i will be used in subsequent figures for an i–punctured Klein bottle and an i–punctured torus respectively, and G_2 for a genus two surface.

The column *class* specifies the equivalence class (defined in Subsection 5.2.4) that the projective normal Q–coordinate V_i of F_i belongs to.

The column *∂–curves* encodes the number of boundary components on the respective cusps; if there are i boundary components on the (red) cusp 0, and j on the (green) cusp 1, this is written as i/j.

Last, the corresponding boundary curves are given as they are computed from the chosen (oriented, topological) peripheral system. The holonomies yield the following quadrilateral combinations:

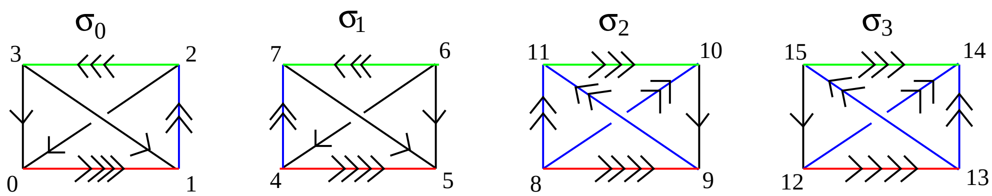
$$\nu(\mathcal{M}_0) = -q_0 + q_0' + q_2 - q_2' - q_3' + q_3'' \qquad \nu(\mathcal{L}_0^t) = -2q_1' + 2q_1'' - 2q_2' + 2q_2''$$
$$\nu(\mathcal{M}_1) = q_0 - q_0' + q_2' - q_2'' - q_3 + q_3' \qquad \nu(\mathcal{L}_1^t) = -2q_0' + 2q_0'' - 2q_2' + 2q_2''$$

The signs in the table respect the transverse orientations, which is relevant when the boundary curves of surfaces corresponding to linear combinations of the $N(F_i)$ are computed.

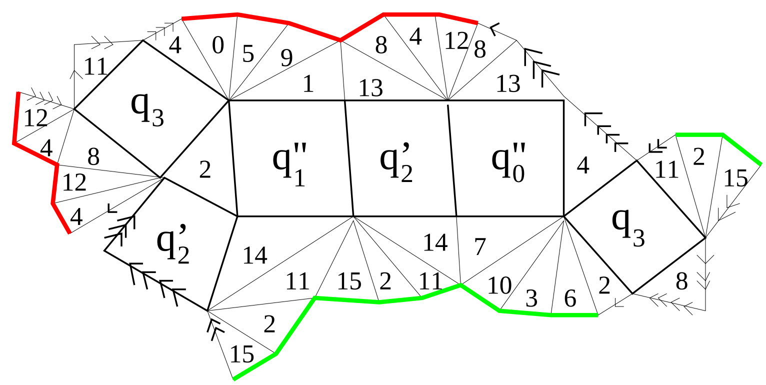
5.2.4. Equivalence classes. Proposition 4.5 provides a homeomorphism $N : \mathfrak{D}_?(\mathcal{W}) \to \mathcal{N}(\mathcal{W})$ between the tentative logarithmic limit set of $\mathfrak{D}(\mathcal{W})$ and $\mathcal{N}(\mathcal{W})$. There is an induced action of the group D_4 of symmetries of $\mathfrak{D}(\mathcal{W})$ on $\mathfrak{D}_?(\mathcal{W})$. Since the elements of D_4 interchange coordinates, the induced action corresponds to interchanging coordinate triples of elements in $\mathfrak{D}_\infty(\mathcal{W})$. Moreover, there is an

Vertex	q_0	q_0'	q_0''	q_1	q_1'	q_1''	q_2	q_2'	q_2''	q_3	q_3'	q_3''	type	class	∂–curves	$\nu(\mathcal{L}_0^t)$,	$-\nu(\mathcal{M}_0)$,	$\nu(\mathcal{L}_1^t)$,	$-\nu(\mathcal{M}_1)$
1	1	0	0	0	0	0	1	0	0	0	0	0	T_1	N_1'	0/1	0,	0,	0,	–1
2	0	0	0	1	0	0	1	0	0	0	0	0	T_1	N_1	1/0	0,	–1,	0,	0
3	0	0	0	1	0	0	0	0	0	1	0	0	T_1	N_1'	0/1	0,	0,	0,	1
4	1	0	0	0	0	0	0	0	0	1	0	0	T_1	N_1	1/0	0,	1,	0,	0
5	1	0	0	0	1	0	0	0	0	0	0	1	S_3	N_2'	2/1	–2,	0,	0,	–1
6	0	0	0	0	0	1	1	0	0	0	1	0	S_3	N_2'	2/1	2,	0,	0,	–1
7	0	0	1	0	0	0	1	0	0	0	1	0	S_3	N_2	1/2	0,	–1,	2,	0
8	0	1	0	1	0	0	0	0	0	0	0	1	S_3	N_2	1/2	0,	–1,	–2,	0
9	0	1	0	1	0	0	0	0	1	0	0	0	S_3	N_2'	2/1	2,	0,	0,	1
10	0	0	1	0	0	0	0	1	0	1	0	0	S_3	N_2'	2/1	–2,	0,	0,	1
11	0	0	0	0	0	1	0	1	0	1	0	0	S_3	N_2	1/2	0,	1,	–2,	0
12	1	0	0	0	1	0	0	0	1	0	0	0	S_3	N_2	1/2	0,	1,	2,	0
13	0	0	0	0	1	0	0	1	0	0	0	2	R_2	N_3'	1/1	–4,	–1,	–2,	–1
14	0	1	0	0	0	2	0	0	0	0	1	0	R_2	N_3'	1/1	4,	1,	–2,	–1
15	0	0	2	0	1	0	0	0	0	0	1	0	R_2	N_3	1/1	–2,	–1,	4,	1
16	0	1	0	0	0	0	0	1	0	0	0	2	R_2	N_3	1/1	–2,	–1,	–4,	–1
17	0	1	0	0	0	0	0	0	2	0	1	0	R_2	N_3'	1/1	4,	1,	2,	1
18	0	0	2	0	1	0	0	1	0	0	0	0	R_2	N_3'	1/1	–4,	–1,	2,	1
19	0	1	0	0	0	2	0	1	0	0	0	0	R_2	N_3	1/1	2,	1,	–4,	–1
20	0	0	0	0	1	0	0	0	2	0	1	0	R_2	N_3	1/1	2,	1,	4,	1

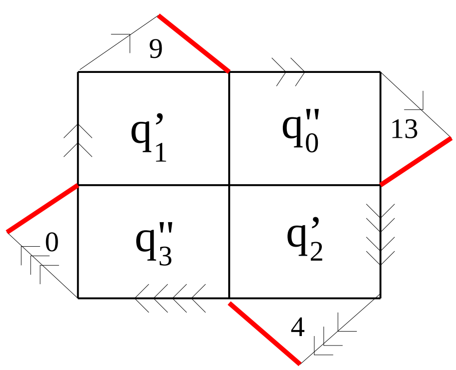
TABLE 5.6. Minimal representatives for vertex solutions



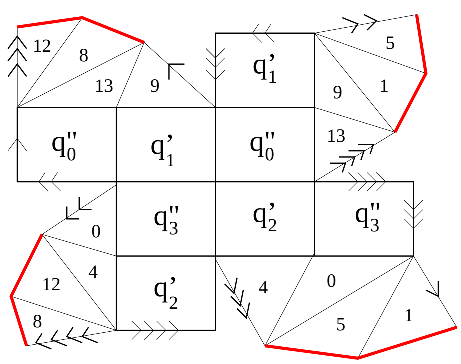
(a)



(b) The twice–punctured Klein bottle $N(F_{10}) + N(F_{11})$

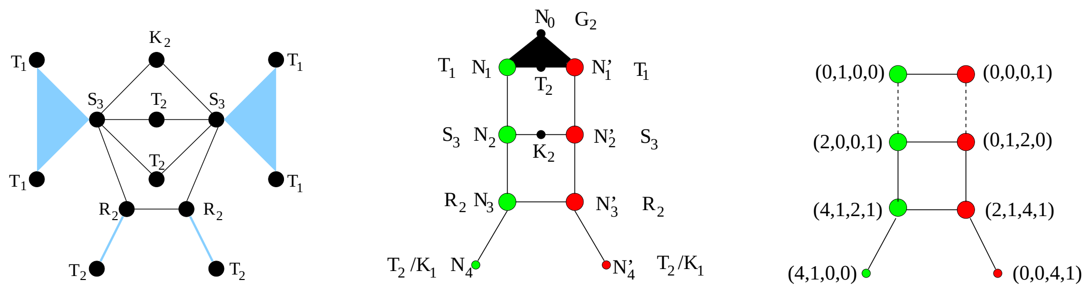


(c) The once–punctured Klein bottle $\frac{1}{2}(N(F_{13}) + N(F_{18}))$



(d) The twice–punctured torus $N(F_{13}) + N(F_{18})$

FIGURE 5.7. Normal surfaces in the Whitehead link complement



(a) Floyd–Hatcher (b) Classes (c) ∂–curves

FIGURE 5.8. Surfaces in the Whitehead link complement

induced action of D_4 on $\mathcal{N}(\mathcal{W})$ via the homeomorphism N which again corresponds to interchanging coordinate triples of elements.

Indeed, D_4 can be identified with a group of symmetries of the triangulation. There is a Kleinian four group K_f of symmetries of the triangulation which stabilises the cusps. K_f can be identified with the group $< \tau_1, \tau_2 >$, and orbits of the action of K_f on $\mathcal{N}(\mathcal{W})$ give six equivalence classes amongst the vertex solutions in $\mathcal{N}(\mathcal{W})$:

$$N_1 = \{V_2, V_4\}, \qquad N_1' = \{V_1, V_3\},$$

$$N_2 = \{V_7, V_8, V_{11}, V_{12}\}, \qquad N_2' = \{V_5, V_6, V_9, V_{10}\},$$

$$N_3 = \{V_{15}, V_{16}, V_{19}, V_{20}\}, \qquad N_3' = \{V_{13}, V_{14}, V_{17}, V_{18}\}.$$

Members of the classes N_i and N_i' are interchanged by symmetries interchanging the cusps. These symmetries correspond to the remaining elements of D_4. One can visualise the action of D_4 on $\mathcal{N}(\mathcal{W})$ by considering the action of the dihedral group on Figure 5.6 induced by its standard action on a square. The quotient by the action of K_f is pictured in Figure 5.8(b), where we also indicate the topological type of some surfaces with the corresponding projective normal Q–coordinates.

Note that there are arcs in $\mathcal{N}(\mathcal{W})$ connecting elements of N_3, e.g. the vertices V_{16} and V_{19}. Taking $N(F_{16})+N(F_{19})$, we obtain a twice–punctured torus. However,

$\frac{1}{2}(N(F_{16}) + N(F_{19}))$ is also an admissible integer solution, and the corresponding normal surface is a once-punctured Klein bottle. We denote the corresponding equivalence class (i.e. K_f orbit) by N_4. Similarly, for arcs in $\mathcal{N}(\mathcal{W})$ joining elements of N'_3, we obtain an equivalence class N'_4 whose elements are midpoints of these arcs.

The surface determined by a minimal integer solution corresponding to the point $V_0 := \frac{1}{2}(V_1 + V_3) = \frac{1}{2}(V_2 + V_4)$ in the centre of $\mathcal{N}(\mathcal{W})$ is a genus two surface. This point is fixed by all symmetries, and we denote its equivalence class by N_0.

We now analyse the D_4 orbit of a point $P \in N(\mathcal{W})$. If P is the centre of the square, i.e. the point $\frac{1}{2}(V_1+V_3)$, then we have already remarked that its equivalence class only contains one element.

If P is contained in the square $[V_1, V_2, V_3, V_4]$ but not equal to its centre, then its D_4 orbit contains exactly four elements. Note that different elements in the square can have the same *∂–coordinate*

$$(\nu_P(\mathcal{L}_0), -\nu_P(\mathcal{M}_0), \nu_P(\mathcal{L}_1), -\nu_P(\mathcal{M}_1)).$$

This is true for instance for V_1 and $\frac{3}{4}V_1 + \frac{1}{4}V_3$. However, elements of the same D_4 orbit are distinguished by their ∂–coordinates.

If P is contained in N_4 or N'_4, or is the midpoint of an arc $[V, W]$ in $\mathcal{N}(\mathcal{W})$ with $V \in N_i$, $W \in N'_i$ and $i = 2$ or 3, then its D_4 orbit contains exactly four elements. Moreover, the elements of the orbit are distinguished by their ∂–coordinates.

If $P \in N(\mathcal{W})$ is not contained in any of the sets considered above, then its D_4 orbit contains exactly eight elements, and all these elements are distinguished by their ∂–coordinates.

Moreover, the elements of an equivalence class (i.e. K_f orbit) of a point in $\mathcal{N}(\mathcal{W})$ have the same projectivised (i.e. unoriented) ∂–coordinate. Thus, the set of projectivised ∂–coordinates arising from $\mathcal{N}(\mathcal{W})$ can be computed using the incidence structure amongst the equivalence classes, and the result is shown in Figure 5.8(c), where all arcs, including the dashed ones, are included. This shows that each equivalence class is uniquely determined by its projectivised ∂–coordinate unless its elements are contained in the centre square of $\mathcal{N}(\mathcal{W})$, and hence yields a stronger statement:

LEMMA 5.5. *Any embedded normal surface in* $\mathcal{W}$ *is uniquely determined by its* transversely oriented *boundary curves unless its projectivised normal* Q*–coordinate is contained in the centre square in* $\mathcal{N}(\mathcal{W})$*.*

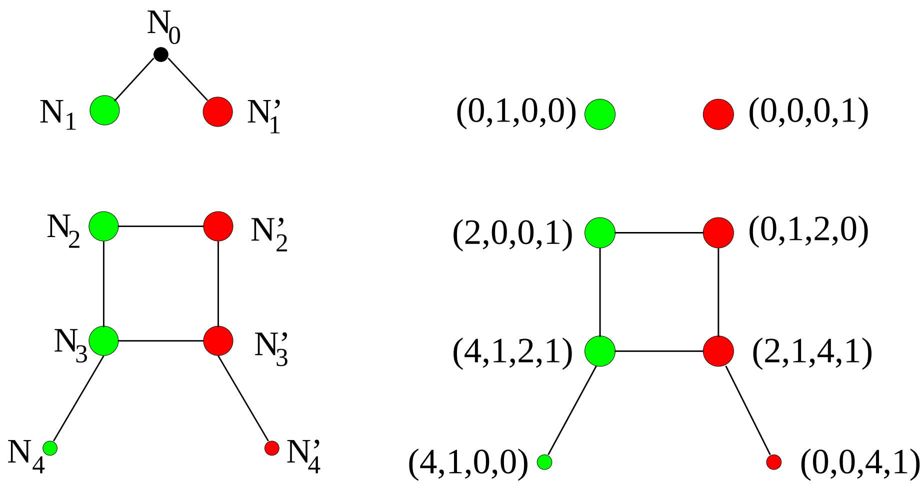
PROOF. Let S be an embedded normal surface. Then there is a unique $\alpha > 0$ and a unique point $P \in \mathcal{N}(\mathcal{W})$ such that $N(S) = \alpha P$. If P is not contained in the centre square, then there is no other point in $\mathcal{N}(\mathcal{W})$ with the same ∂–coordinate. Thus, S is uniquely determined by

$$\begin{aligned} &\alpha(\nu_P(\mathcal{L}_0), -\nu_P(\mathcal{M}_0), \nu_P(\mathcal{L}_1), -\nu_P(\mathcal{M}_1)) \\ =&(\nu_{N(S)}(\mathcal{L}_0), -\nu_{N(S)}(\mathcal{M}_0), \nu_{N(S)}(\mathcal{L}_1), -\nu_{N(S)}(\mathcal{M}_1)). \end{aligned}$$

This proves the lemma. ■

5.2.5. Incompressible surfaces. The incompressible surfaces in $\mathcal{W}$ are classified in [**21**], and their result is shown in Figure 5.8(a). The blue areas and arcs correspond to surfaces obtained from disjoint unions, whilst the black arcs arise from a procedure called *pinching*. The structure of this space is somewhat different to the structure of the set of equivalence classes of $\mathcal{N}$. One might be able to find a normal surface space more closely resembling the space of [**21**] by subdividing the triangulation. However, this is not done here. The vertices of $\mathcal{N}(\mathcal{W})$ and combinations thereof can be related to the surfaces given in [**21**] by comparing the boundary curves and topological types, as well as the position in the manifold.

5.2.6. Boundary curve space. Lash computes the boundary curves arising from incompressible (and hence ∂–incompressible) surfaces in [**32**], which implies that the dashed arcs in Figure 5.8(c) do not arise from incompressible surfaces.



(a) Detected surfaces (b) Detected boundary curves

FIGURE 5.9. The logarithmic limit sets of $\mathfrak{D}(\mathcal{W})$ and $\overline{\mathfrak{E}}_0(\mathcal{W})$ modulo $\mathbb{Z}_2^2$

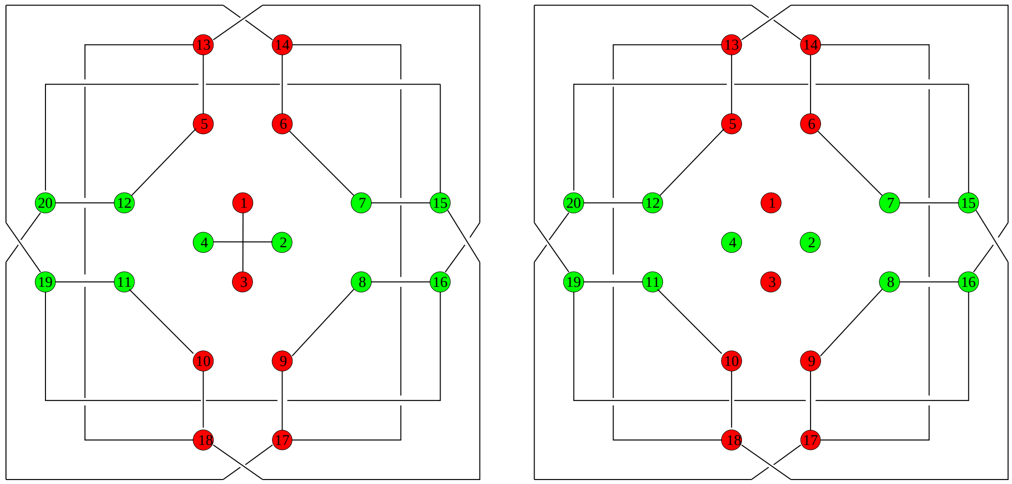
The remaining set is identical to Lash's, taking into account that he works with a left–handed link projection.

The boundary curve space has two connected components — one is an interval, and the other is a circle with two intervals attached to it. Lash shows that the former arises from reducible $SL_2(\mathbb{C})$–representations, and that the latter arises from irreducible $SL_2(\mathbb{C})$–representations.

5.3. Surfaces arising from degenerations

We have a homeomorphism $N : \mathfrak{D}_?(\mathcal{W}) \to \mathcal{N}(\mathcal{W})$ between the tentative logarithmic limit set and the projective admissible solution space. We now determine $N(\mathfrak{D}_\infty(\mathcal{W}))$ and hence $\mathfrak{D}_\infty(\mathcal{W})$. This analysis together with Proposition 4.6 also determines the logarithmic limit set of $\overline{\mathfrak{E}}_0(\mathcal{W})$.

The main result of this section is summarised in Figures 5.9 and 5.10. We first show that all vertices of $\mathcal{N}(\mathcal{W})$ are contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$. We then determine which parts of the centre square are contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$. For the remaining



(a) Logarithmic limit set of $\mathfrak{D}(\mathcal{W})$ (b) Logarithmic limit set of $\overline{\mathfrak{E}}_0(\mathcal{W})$

FIGURE 5.10. The logarithmic limit sets of $\mathfrak{D}(\mathcal{W})$ and $\overline{\mathfrak{E}}_0(\mathcal{W})$

arcs, we use the birational equivalence $\Psi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{E}}_0(\mathcal{W})$, Lemma 5.5 and Lash's result.

The computation is simplified by the following observation. The action of D_4 on $\mathfrak{D}_?(\mathcal{W})$ stabilises $\mathfrak{D}_\infty(\mathcal{W})$. It follows that if some point in $\mathcal{N}(\mathcal{W})$ has preimage in $\mathfrak{D}_\infty(\mathcal{W})$, so has its orbit under the D_4 action on $\mathcal{N}(\mathcal{W})$. It will also be convenient to work with integer valued valuations, and hence we often rescale points in $\mathcal{N}(\mathcal{W})$ and $\mathfrak{D}_\infty(\mathcal{W})$ to integer coordinates, and we say that a normal surface S is *detected* by $\xi \in \mathfrak{D}_\infty(\mathcal{W})$ if $N(S) = \alpha N(\xi)$ for some $\alpha > 0$.

5.3.1. Detected vertices. In order to show that all vertices of $\mathcal{N}(\mathcal{W})$ correspond to points in $\mathfrak{D}_\infty(\mathcal{W})$, it is sufficient to show this for any one point of the first three sets below. The remaining sets expand this program to the vertices of Figure 5.8(c).

1. $N_1 \cup N_1' = \{V_1, \ldots, V_4\}$
2. $N_2 \cup N_2' = \{V_5, \ldots, V_{12}\}$
3. $N_3 \cup N_3' = \{V_{13}, \ldots, V_{20}\}$

4. $\{V_0 = \frac{1}{2}(V_1 + V_3) = \frac{1}{2}(V_2 + V_4)\}$
5. $\{\frac{1}{2}(V_{13} + V_{18}), \frac{1}{2}(V_{14} + V_{17}), \frac{1}{2}(V_{15} + V_{20}), \frac{1}{2}(V_{16} + V_{19})\}$

Recall that $\mathfrak{D}'(\mathcal{W})$ is defined by

$$1 = wxyz \quad \text{and} \quad wx(1-y)(1-z) = (1-w)(1-x)yz, \tag{5.26}$$

and that for any $w, x, y, z \in \mathbb{C} - \{0, 1\}$ subject to these equations, there is a unique point on $\mathfrak{D}(\mathcal{W})$. We will now describe sequences of points in $\mathfrak{D}'(\mathcal{W})$ which correspond to well–defined sequences in $\mathfrak{D}(\mathcal{W})$ approaching the desired ideal points.

1. The point $(w, x, y, z) = (w, -w^{-1}, w, -w^{-1})$ satisfies equations (5.26) for any $w \in \mathbb{C} - \{0\}$, and if w has positive imaginary part, it defines a positively oriented triangulation of $\mathcal{W}$, and in particular an incomplete hyperbolic structure on $\mathcal{W}$. This implies that the triangulation can be deformed through positively oriented tetrahedra from the complete structure where $w = i$ to the ideal points where $w \to 0$ or $w \to \pm 1$. These points correspond to V_1 and V_3 for $w \to 1$ and $w \to -1$ respectively, and to $\frac{1}{2}(V_{16}+V_{19})$ for $w \to 0$, since the growth rates of all parameters are equal.

We give one example, where we scale to integer coordinates. As $w \to 0$, the ideal point

$$\xi = (-1, 0, 1, 1, -1, 0, -1, 0, 1, 1, -1, 0)$$

is approached. The corresponding normal Q–coordinate is

$$N(\xi) = (0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1),$$

$$\text{whilst} \quad \frac{1}{2}(N(S_{16}) + N(S_{19})) = (0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1),$$

which we can rescale to $\frac{1}{2}(V_{16} + V_{19})$.

Moreover, one easily obtains the limiting eigenvalues. As $w \to 0$ there are the following power series expansions for the resulting eigenvalues:

$$\overline{s} = 1, \qquad t = -1,$$
$$\overline{u} = \frac{1+w}{w(1-w)} = \frac{1}{w} + 2\sum_{i=0}^{\infty} w^i, \qquad v = w^2.$$

The first two equations reflect the fact that one cusp is completed, and comparing the growth rates at the second cusp (remembering that $\overline{u}$ is the square of an eigenvalue, whilst v is an eigenvalue) gives a point in boundary curve space with coordinates $[0, 0, 4, 1]$.

2. For any $w \in \mathbb{C} - \{0, \pm 1\}$, the point

$$(w, x, y, z) = (w, \frac{1-w}{1+w}, -\frac{1+w}{1-w}, -w^{-1}) \in \mathfrak{D}'(\mathcal{W})$$

gives a point on $\mathfrak{D}(\mathcal{W})$. Note that the associated triangulation involves either only flat tetrahedra or both positively and negatively oriented ones. One may use power series expansions to show that a detected surface for $w \to 0$ is S_8:

$$x = \frac{1-w}{1+w} = 1 + 2\sum_{i=1}^{\infty}(-1)^i w^i, \quad y = -\frac{1+w}{1-w} = -1 - 2\sum_{i=1}^{\infty} w^i, \quad z = -\frac{1}{w}.$$

Thus, as $w \to 0$, the ideal point

$$\xi = (-1, 0, 1, 0, 1, -1, 0, 0, 0, 1, -1, 0)$$

is approached. The corresponding normal Q–coordinate is

$$N(\xi) = (0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1),$$

which coincides with the normal Q–coordinate of S_8, and may be rescaled to V_8.

3. Detecting elements of $N_3 \cup N_3'$ is a little more involved. Let

$$w = \frac{-1 - \epsilon^3 + \sqrt{5 - 4\epsilon + 4\epsilon^2 - 2\epsilon^3 + \epsilon^6}}{2(1+\epsilon^2)} \qquad x = \epsilon$$
$$y = -\epsilon \frac{2(1+\epsilon^2)}{-1 - \epsilon^3 + \sqrt{5 - 4\epsilon + 4\epsilon^2 - 2\epsilon^3 + \epsilon^6}} \qquad z = -\epsilon^{-2}$$

Formal substitution of these assignments in the defining equations of the deformation variety shows that whenever the expressions are defined for any small, non–zero ϵ, they determine a point of $\mathfrak{D}(\mathcal{W})$. Since w is differentiable at $\epsilon = 0$ and takes the value $w(0) = \frac{-1\pm\sqrt{5}}{2}$, it follows that w and y have converging power series expansions at $\epsilon = 0$, and y has a zero of order one at $\epsilon = 0$. The first few terms are:

$$w(\epsilon) = \frac{-1\pm\sqrt{5}}{2} - \frac{1}{\sqrt{5}}\epsilon + \big(1 - \frac{17}{5\sqrt{5}}\big)\epsilon^2 + \big(-3 + \frac{123}{25\sqrt{5}}\big)\epsilon^3 + \dots$$
$$y(\epsilon) = -\frac{2}{-1+\sqrt{5}}\epsilon - \frac{8}{\sqrt{5}(-1+\sqrt{5})^2}\epsilon^2 + \dots$$

Thus, as $w \to 0$, we approach

$$\xi = (0,0,0,-1,0,1,-1,0,1,2,-2,0).$$

A detected surface is therefore S_{13}.

4. Let $(w,x,y,z) = (w, w^{-1}, w^{-1}, w)$. Then $w \to 1$ approaches an ideal point corresponding to V_0. The chosen degeneration is through triangulations involving positively and negatively oriented tetrahedra, or triangulations which are entirely flat. In fact, there cannot be a degeneration through positively oriented tetrahedra to this ideal point, since the hyperbolic gluing equation $1 = wxyz$ would imply that throughout this degeneration $\arg(w) + \arg(x) + \arg(y) + \arg(z) = 2\pi$, whilst for each parameter the argument converges to zero (see the proof of Proposition 4.4).

5. See point 1. above.

5.3.2. Experimentation with `SnapPea` suggests that elements from all but the fourth set can be approached through degenerations only involving positively oriented tetrahedra. Using the projection of the right–handed Whitehead link, the following surgery coefficients can be approached through degenerations involving positively oriented triangulations:

- $(\infty), (4,1)$. Four tetrahedra degenerate and the splitting surface is a 1–sided Klein bottle.
- $(\infty), (0,1)$. Two tetrahedra degenerate and two become flat with parameters equal to -1. The splitting surface is a 2–sided torus.
- $(4,0), (0,2)$. Three tetrahedra degenerate, the remaining tetrahedron has shape $\frac{1}{2}+\frac{1}{2}i$. The limiting orbifold has volume approximately 0.9159.
- $(8,2), (4,2)$. Three tetrahedra degenerate, the remaining tetrahedron has shape $1+i$, and the limiting orbifold has volume approximately 0.9159.

(`SnapPea` does not compute a splitting surface if all cusps have been filled.)

5.3.3. Centre square. We have already shown that the vertices of the square as well as its centre correspond to ideal points of $\mathfrak{D}(\mathcal{W})$. Any point in the interior of an arc in the boundary of the square corresponds to an ideal point where three of w, x, y, z tend to one, but one of them tends to a complex number other than zero or one. But the relation $wxyz = 1$ does not allow this.

Thus, we now only need to consider points in the interior of the square. Recall the third description of the logarithmic limit set of Subsection 3.1.1. Let J be the defining ideal of $\mathfrak{D}(\mathcal{W})$. Then $\xi \in \mathfrak{D}_\infty(\mathcal{W})$ if for all non–zero $f \in J$, the maximum value of the dot product $\xi \cdot \alpha$ as α runs over the support $s(f)$ is assumed at least twice.

Substitute the expressions $w = \frac{w'-1}{w'}$, $w'' = \frac{1}{1-w'}$, etc., into the elements $wxyz-1$ and $w''x'' - y''z''$ of J (see (5.1)) and adjust by units to obtain the following two elements of $J \cap \mathbb{C}[w', x', y', z']$:

$$\begin{aligned} g_1 &= -w'x'y' - w'x'z' + w'x' - w'y'z' + w'y' + w'z' - w' \\ &\qquad - x'y'z' + x'y' + x'z' - x' + y'z' - y' - z' + 1, \\ g_2 &= w'x' - w' - x' - y'z' + y' + z'. \end{aligned}$$

Now consider

$$g_3 = g_1 + y'g_2 = -w'x'z' + w'x' - w'y'z' + w'z' - w' - x'y'z' + x'z' - x' - (y')^2z' + (y')^2 + 2y'z' - y' - z' + 1 \in J.$$

Without loss of generality, we may consider the point $p'V_1 + q'V_2 + r'V_3$, where $p', q', r' > 0$, in the interior of $[V_1, V_2, V_3, V_4]$. With suitable $p, q, r > 0$, the corresponding point in $\mathfrak{D}_?(\mathcal{W})$ is:

$$\xi = (0, p, -p, 0, q + r, -q - r, 0, p + q, -p - q, 0, r, -r).$$

The set of $\xi \cdot \alpha$ as α runs over $s(g_3)$ is:

$$\{p + q + 2r, p + q + r, 2p + q + r, p + r, p, p + 2q + 2r, q + 2r, q + r, 2p + 2q + r, 2p + 2q, p + q + r, p + q, r, 0\}$$

Since $p, q, r > 0$, each of the remaining elements of the above set is strictly less than $p+2q+2r$ or $2p+2q+r$. Thus, $\xi \in \mathfrak{D}_\infty(\mathcal{W})$ only if $p+2q+2r = 2p+2q+r$. Hence, $r = p$. But then $r' = p'$, and we have

$$p'V_1 + q'V_2 + p'V_3 = p'(V_1 + V_3) + q'V_2 = p'(V_2 + V_4) + q'V_2 = (p' + q')V_2 + p'V_4,$$

and hence the point is contained on one of the diagonals. We conclude that the square minus the diagonals $[V_1, V_3]$ and $[V_2, V_4]$ is contained in $\mathcal{N}(\mathcal{W}) - N(\mathfrak{D}_\infty(\mathcal{W}))$.

To deal with the diagonals, it is sufficient to consider points in the interior of the arc $[V_0, V_2]$. Such a point is of the form

$$\xi = (0, p, -p, 0, q, -q, 0, q, -q, 0, p, -p) \quad \text{where } p > q > 0.$$

The boundary curve contributed by this point is $(0, p - q, 0, 0)$. We are seeking a sequence of points on $\mathfrak{D}'(\mathcal{W})$ converging to $(w, x, y, z) = (1, 1, 1, 1)$ with the specified growth rates, such that none of the elements in the sequence are contained in the union of hypersurfaces where one of the coordinates is equal to one. This gives a corresponding sequence of points on $\mathfrak{D}(\mathcal{W})$ converging to ξ. Note that we can eliminate w from the system

$$0 = 1 - wxyz, \qquad 0 = wx(1-y)(1-z) - (1-w)(1-x)yz,$$

giving a single equation

$$0 = g(x,y,z) = x - xy - xz + yz - xy^2z^2 + x^2y^2z^2.$$

Let

$$x(\epsilon) = 1 - \epsilon^q, \qquad y(\epsilon) = 1 + \epsilon^q + \epsilon^q Y, \qquad z(\epsilon) = 1 - \epsilon^p.$$

We are seeking a power series expansion of $Y = Y(\epsilon)$ such that $Y(0) = 0$ and:

$$0 = g(x(\epsilon), y(\epsilon), z(\epsilon)) \text{ for all } \epsilon \in B_\delta(0) \text{ for some } \delta > 0.$$

If we find this, then we are done, since substituting $w = (xyz)^{-1}$ gives a suitable sequence on $\mathfrak{D}(M)$ converging to ξ. Put $p = q + k$, where $k > 0$ and consider:

$$\begin{aligned} F(\epsilon, Y) &= \epsilon^{-2q} g(x(\epsilon), y(\epsilon), z(\epsilon)) \\ &= -Y + \epsilon^q + \epsilon^{2q} + 2\epsilon^{k+q} - \epsilon^{2k+q} - 2\epsilon^{k+2q} - \epsilon^{2k+2q} - 2\epsilon^{k+3q} + \epsilon^{2k+3q} \\ &\quad + \epsilon^{2k+4q} - \epsilon^k Y + 2\epsilon^{2q} Y + 4\epsilon^{k+q} Y - 2\epsilon^{2k+2q} Y - 4\epsilon^{k+3q} Y \\ &\quad + 2\epsilon^{2k+4q} Y - \epsilon^q Y^2 + \epsilon^{2q} Y^2 + 2\epsilon^{k+2q} Y^2 - 2\epsilon^{k+3q} Y^2 \\ &\quad - \epsilon^{2k+3q} Y^2 + \epsilon^{2k+4q} Y^2. \end{aligned}$$

Then $F(0,0) = 0$, and we have

$$\begin{aligned} \frac{\partial}{\partial Y} F(\epsilon, Y) &= -1 - \epsilon^k + 2\epsilon^{2q} + 4\epsilon^{k+q} - 2\epsilon^{2k+2q} - 4\epsilon^{k+3q} + 2\epsilon^{2k+4q} \\ &\quad - 2\epsilon^q Y + 2\epsilon^{2q} Y + 4\epsilon^{k+2q} Y - 4\epsilon^{k+3q} Y - 2\epsilon^{2k+3q} Y + 2\epsilon^{2k+4q} Y. \end{aligned}$$

Since $k > 0$ and $q > 0$, we have $\frac{\partial}{\partial Y} F(0,0) = -1$. It follows from the implicit function theorem, that there is a unique convergent power series $\varphi(\epsilon)$ such that

$\varphi(0) = 0$ and such that $F(\epsilon, \varphi(\epsilon)) = 0$. Thus, any element of $[V_0, V_2]$ is contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$.

5.3.4. Detected arcs. Consider the arcs in Figure 5.8(b). We will show that a (and hence any) element of an equivalence class in $[N_4, N_3]$, $[N_3, N_3']$, $[N_2, N_3]$, $[N_2, N_2']$ is contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$, and that a (and hence any) element of an equivalence class in the interior of $[N_1, N_2]$ and $[N_1, N_1']$ is contained in $\mathcal{N}(\mathcal{W}) - N(\mathfrak{D}_\infty(\mathcal{W}))$.

If $V \in N(\mathfrak{D}_\infty(\mathcal{W}))$ with non–zero ∂–coordinate, then its projective ∂–coordinate is strongly detected by $\overline{\mathfrak{E}}_0(\mathcal{W})$, since the map $\Psi : \mathfrak{D}(\mathcal{W}) \to \overline{\mathfrak{E}}_0(\mathcal{W})$ of Subsection 5.1.12 is defined everywhere and Proposition 4.6 applies.

Conversely, the birational inverse $\overline{\mathfrak{E}}_0(\mathcal{W}) \to \mathfrak{D}(\mathcal{W})$ yields that if a projective boundary curve is detected by an ideal point ξ of the $PSL_2(\mathbb{C})$–eigenvalue variety, then it must be the projectivised boundary curve of a normal surface detected by an ideal point of $\mathfrak{D}(\mathcal{W})$, unless ξ is an isolated point in the logarithmic limit set of $\overline{\mathfrak{E}}_0(\mathcal{W})$ and it is an ideal point of the 1–dimensional subvariety (5.23) of $\overline{\mathfrak{E}}_0(\mathcal{W})$ where Ψ^{-1} is not defined. Ideal points of this subvariety detect boundary curves of the form (p, q, p, q), since Ψ^{-1} is not defined on $\overline{s} = \overline{u}, t = v$.

Since the quotient map $\mathrm{q}_3 : \mathfrak{E}_0(\mathcal{W}) \to \overline{\mathfrak{E}}_0(\mathcal{W})$ has the form $(s, t, u, v) \to (s^2, t, u^2, v)$, all boundary curves which are detected by $SL_2(\mathbb{C})$–representations are also detected by $PSL_2(\mathbb{C})$–representations Thus, all boundary slopes arising from irreducible representations into $SL_2(\mathbb{C})$ are detected by the deformation variety possibly apart from the ones which are of the form (p, q, p, q).

Let V be a point of $\mathcal{N}(\mathcal{W})$ which is not contained in the centre square. If the projective ∂–coordinate of V is strongly detected by the deformation variety, then either V or one of the points in its D_4 orbit must be contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$. But this implies that all points in the orbit are contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$. Thus, for equivalence classes outside the centre square, it is enough to determine whether or not their projective boundary curves are detected by $\mathfrak{D}(\mathcal{W})$.

Thus, all elements of the equivalence classes in $[N_2, N_3]$, $[N_3, N_4]$, $[N_2', N_3']$ and $[N_3', N_4']$ are contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$, as well as the elements in $[N_3, N_3']$ and $[N_2, N_2']$ with their midpoints removed (because they have projectivised ∂–coordinates (p, q, p, q)). But since the logarithmic limit set is a union of convex rational polytopes, it follows that the midpoints must be included as well.

The points of $\mathcal{N}(\mathcal{W})$ corresponding to the interior of the dashed arcs in Figure 5.8(c) cannot be contained in $N(\mathfrak{D}_\infty(\mathcal{W}))$, since otherwise there would be incompressible surfaces with these boundary slopes contradicting [**32**]. This shows that the equivalence classes in the interior of $[N_1, N_2]$ and $[N_1', N_2']$ are contained in $\mathcal{N}(\mathcal{W}) - N(\mathfrak{D}_\infty(\mathcal{W}))$.

This completes the computation of $\mathfrak{D}_\infty(\mathcal{W})$.

5.3.5. Logarithmic limit set of the eigenvalue variety. The relationship between ideal points of the $PSL_2(\mathbb{C})$–eigenvalue variety and ideal points of the deformation variety has already been discussed and used in the computation of $\mathfrak{D}_\infty(\mathcal{W})$, and it follows that the logarithmic limit set of $\overline{\mathfrak{E}}_0(\mathcal{W})$ is as pictured in Figure 5.10(b). In particular, all these slopes arise from sequences of irreducible representation. For all but four points this follows also from [**32**].

The sequences in Subsection 5.3.1 under point 1. which converge to ideal points corresponding to elements of N_1 and N_1' give rise to sequences of irreducible representations in the character variety since the eigenvalue of a longitude is constant equal to -1 throughout the degeneration. There also are sequences of reducible representations detecting the same slopes:

$$\rho_m(\mathcal{M}_i) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho_m(\mathcal{M}_j) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix},$$

where $m \to 0$ or $m \to \infty$.

The eigenvalue variety is irreducible because it is birationally equivalent to $\mathfrak{D}(\mathcal{W})$ (see also Subsection 5.4.3). Bergman asks in [**3**] whether the logarithmic limit set of an irreducible variety V is always a union of polytopes *all* having (real) dimension $\dim_{\mathbb{C}} V - 1$. We now know that this is not so in the case of the eigenvalue variety of the Whitehead link. It would be interesting to modify Bergman's question and ask whether each connected component of the logarithmic limit set of an irreducible variety can always be expressed as a union of polytopes all having the same dimension when $\dim_{\mathbb{C}} V > 2$.

5.4. Calculations

5.4.1. Manifold data. The "gluing and completeness" data obtained from `snap` is given in Table 5.7. The relationship to the shape parameters used here is: $w = z_1'$, $x = z_2''$, $y = z_3''$ and $z = z_4''$. From this, one can verify that the holonomies of the peripheral elements given by `snap` are the inverses of the holonomies of the meridians and geometric longitudes given above, where `snap`'s cusp 0 corresponds to the green cusp (here: cusp 1), and cusp 1 to the red cusp (here: cusp 0).

	z_1	z_2	z_3	z_4	$1-z_1$	$1-z_2$	$1-z_3$	$1-z_4$	
$H'(\mathcal{M}_0)$	1	0	–1	0	–1	0	1	1	0
$H'(\mathcal{M}_1)$	0	0	0	1	1	–1	0	0	0
$H'(\mathcal{L}_0)$	1	0	0	0	0	1	–1	1	0
$H'(\mathcal{L}_1)$	–1	0	0	0	0	–1	–1	1	0
e_1	1	1	1	1	1	–2	0	0	–1
e_2	0	–1	–1	–1	–1	1	1	1	1
e_3	–1	1	1	1	1	0	–2	–2	–1
e_4	0	–1	–1	–1	–1	1	1	1	1

TABLE 5.7. `snap`'s gluing matrix for $m129$

5.4.2. Elimination theory. The calculations below use the elimination and extension theorems as found in [**16**]. Given two polynomials $f_1, f_2 \in \mathbb{C}[x_1, \ldots, x_m]$ of positive degree in x_1, let $g_1, g_2 \in \mathbb{C}[x_2, \ldots, x_m]$ such that $f_i = g_i x_1^{n_i} +$ terms in which x_1 has degree lower than n_i. The resultant $Res(f_1, f_2, x_1)$ is an element of $\mathbb{C}[x_2, \ldots, x_m]$. If this resultant vanishes at $P = (p_2, \ldots, p_m)$ and not both of g_1 and

g_2 vanish at P, then there is $p_1 \in \mathbb{C}$ such that f_1 and f_2 vanish at $(p_1, \ldots, p_m)$. This fact can iteratively be applied to collections of polynomials, and to eliminate more than one variable.

5.4.3. Eigenvalue variety. Denote the component in $\mathfrak{E}(\mathcal{W})$ arising from reducible representations by $\mathfrak{E}^r(\mathcal{W})$. We have $\mathfrak{E}_0(\mathcal{W}) = \overline{\mathfrak{E}(\mathcal{W}) - \mathfrak{E}^r(\mathcal{W})}$, where the overline denotes the Zariski closure. With respect to the chosen affine coordinates, we have $\mathfrak{E}^r(\mathcal{W}) = \{t = v = 1\}$, which is 2–dimensional. $\mathfrak{E}_0(\mathcal{W})$ is 2–dimensional since it is the closure of the image of the eigenvalue map. From this, it also follows that the intersection of $\mathfrak{E}_0(\mathcal{W})$ and $\mathfrak{E}^r(\mathcal{W})$ is a union of two lines:

$$\mathfrak{E}^r(\mathcal{W}) \cap \mathfrak{E}_0(\mathcal{W}) = \{s = t = v = 1\} \cup \{t = u = v = 1\}.$$

Consider $\mathfrak{C}(\mathcal{W})$ and the eigenvalue map e (5.18). The rational functions determining the eigenvalues of the longitudes are:

$$t = \vartheta_{\mathcal{L}_0^t}(\rho) = s^{-2} - s^{-2}u^2 + u^2 + c(2s^{-1}u + s^{-3}u^{-1} - s^{-3}u) + c^2 s^{-2},$$
$$v = \vartheta_{\mathcal{L}_1^t}(\rho) = u^{-2} - u^{-2}s^2 + s^2 + c(2u^{-1}s + u^{-3}s^{-1} - u^{-3}s) + c^2 u^{-2}.$$

Then $s^2 t - u^2 v = s^2 - u^2 + c(su^{-1} - s^{-1}u)$. Since $\mathfrak{C}(\mathcal{W})$ is not contained in the hyperplane $s^2 = u^2$, this shows that the map $\mathrm{e} : \mathfrak{C}(\mathcal{W}) \to \mathfrak{E}_0(\mathcal{W})$ has degree one, and also determines an inverse mapping:

$$(s, t, u, v) \to \left(s, u, \frac{s^2(t-1) + u^2(1-v)}{su^{-1} - s^{-1}u} \right). \tag{5.27}$$

Recall the defining equation (5.12) of $\mathfrak{C}(\mathcal{W})$:

$$f_1 = (s - s^{-1})(u - u^{-1}) + c(s^{-2}u^{-2} - u^{-2} - s^{-2} + 4 - s^2 - u^2 + s^2u^2)$$
$$+ c^2(2s^{-1}u^{-1} - su^{-1} - s^{-1}u + 2su) + c^3.$$

We obtain two additional polynomials $f_2 = s^3ut + \ldots$ and $f_3 = su^3v + \ldots$ from the above expressions for t and v. The only variable to be eliminated is c, and the leading coefficients of c in f_1, f_2 and f_3 are monomials in $\mathbb{C}[s, u]$. Since s and u are units, it follows that the eigenvalue variety is defined by $Res(f_1, f_2, c) = Res(f_1, f_3, c) =$

$Res(f_2, f_3, c) = 0$. Eliminating redundant factors from the resultants gives the following set of defining equations for $\mathfrak{E}_0(\mathcal{W})$:

$$\begin{aligned} h_1 &= t - s^2t + s^2t^2 - s^4t^2 - u^2 - 2s^2tu^2 + s^4tu^2 - t^2u^2 + 2s^2t^2u^2 \\ &\quad + s^4t^3u^2 + tu^4 - s^2tu^4 + s^2t^2u^4 - s^4t^2u^4, \\ h_2 &= s^2 - v - s^4v + u^2v + 2s^2u^2v + s^4u^2v - s^2u^4v + s^2v^2 - u^2v^2 \\ &\quad - 2s^2u^2v^2 - s^4u^2v^2 + u^4v^2 + s^4u^4v^2 - s^2u^4v^3, \\ h_3 &= s^4t - s^6t - s^2tu^2 + s^4tu^2 + s^6t^2u^2 - s^2u^2v \\ &\quad + u^4v + s^2u^4v - 2s^4tu^4v - u^6v + s^2u^6v^2. \end{aligned}$$

The mapping (5.27) is not defined when $s^2 = u^2$. We may now compute the defining equations of the subvariety of $\mathfrak{E}_0(\mathcal{W})$ on which it is not defined:

$$s^2 = u^2, \qquad t = v, \qquad 0 = -t + s^2(1+t) + s^4t(1-t) - s^6t^2(1+t) + s^8t^2.$$

With a little more effort, one can compute the following inverse mappings defined on two open sets, which cover all but eight points of the eigenvalue variety:

$$\text{(5.28)} \qquad \mathrm{e}_1^{-1} : \mathfrak{E}^i(\mathcal{W}) \to \mathfrak{C}(\mathcal{W}) \qquad (s,t,u,v) \to \left(s, u, \frac{(s^2-v)(1-u^2)}{su(1+v)} \right)$$

$$\text{(5.29)} \qquad \mathrm{e}_2^{-1} : \mathfrak{E}^i(\mathcal{W}) \to \mathfrak{C}(\mathcal{W}) \qquad (s,t,u,v) \to \left(s, u, \frac{(u^2-t)(1-s^2)}{su(1+t)} \right)$$

The maps reflect the interchangeability of the two cusps. The points corresponding to the complete structure are always singularities of the eigenvalue variety — here determined by the points where $t = v = -1$ and $s^2 = u^2 = 1$. The other points of $\mathfrak{E}_0(\mathcal{W})$ where neither of the above maps are defined are subject to $t = v = -1$ and $s^2 = u^2 = -1$.

5.4.4. The defining equations for the $PSL_2(\mathbb{C})$–eigenvalue variety can be worked out from $\overline{\mathfrak{C}}(\mathcal{W})$ similarly to the above, or from the results of the previous subsection using Lemma 5.3. In particular, we obtain:

$$
\begin{aligned}
t =& \vartheta_{\mathcal{L}_0^t}(\overline{\rho}_{GL}) = \overline{s}^{-1} - \overline{s}^{-1}\overline{u} + \overline{u} + d(2\overline{s}^{-1} + \overline{s}^{-2}\overline{u}^{-1} - \overline{s}^{-2}) + d^2\overline{s}^{-2}\overline{u}^{-1},\\
v =& \vartheta_{\mathcal{L}_1^t}(\overline{\rho}_{GL}) = \overline{u}^{-1} - \overline{u}^{-1}\overline{s} + \overline{s} + d(2\overline{u}^{-1} + \overline{u}^{-2}\overline{s}^{-1} - \overline{u}^{-2}) + d^2\overline{u}^{-2}\overline{s}^{-1}.
\end{aligned}
$$

Then $\overline{s}\overline{u}^2 v - \overline{s}^2\overline{u}t = \overline{s}\overline{u}^2 - \overline{s}^2\overline{u} + d(\overline{u} - \overline{s})$ implies that the degree of the map $\overline{\mathfrak{C}}(\mathcal{W}) \to \overline{\mathfrak{E}}(\mathcal{W})$ is equal to one. It follows from the above discussion that there are only two points where neither of the following inverse maps is not defined:

$$
\overline{\mathrm{e}}_1^{-1} : \overline{\mathfrak{E}}_0(\mathcal{W}) \to \overline{\mathfrak{C}}(\mathcal{W}) \qquad (\overline{s}, t, \overline{u}, v) \to \left(\overline{s}, \overline{u}, \frac{(\overline{s} - v)(1 - \overline{u})}{(1 + v)}\right) \tag{5.30}
$$

$$
\overline{\mathrm{e}}_2^{-1} : \overline{\mathfrak{E}}_0(\mathcal{W}) \to \overline{\mathfrak{C}}(\mathcal{W}) \qquad (\overline{s}, t, \overline{u}, v) \to \left(\overline{s}, \overline{u}, \frac{(\overline{u} - t)(1 - \overline{s})}{(1 + t)}\right). \tag{5.31}
$$

The results of this and the previous subsections are summarised in the following lemma. Note that this provides an alternative proof of Lemma 5.4.

LEMMA 5.6. *The varieties* $\mathfrak{C}(\mathcal{W})$ *and* $\mathfrak{E}_0(\mathcal{W})$ *are birationally equivalent, and so are the varieties* $\overline{\mathfrak{C}}(\mathcal{W})$ *and* $\overline{\mathfrak{E}}_0(\mathcal{W})$.

5.4.5. If one of the cusps is assumed to be complete, then the resulting subvariety, which parametrises hyperbolic Dehn fillings on the other cusp, is defined by:

$$
0 = 1 - t + 4\overline{s}t - \overline{s}^2 t + \overline{s}^2 t^2 \qquad \text{when } \overline{u} = 1, v = -1, \tag{5.32}
$$

$$
0 = 1 - v + 4\overline{u}v - \overline{u}^2 v + \overline{u}^2 v^2 \qquad \text{when } \overline{s} = 1, t = -1. \tag{5.33}
$$

The boundary curves $(0,0,0,1)$, $(0,0,4,1)$ and $(0,1,0,0)$, $(4,1,0,0)$ are detected by these curves respectively. We may also write the defining equations as the following well–defined trace relations:

$$
4 = \operatorname{tr}\overline{\rho}(\mathcal{M}_i^2) - \operatorname{tr}\overline{\rho}(\mathcal{M}_i^2\mathcal{L}_i^t) = \operatorname{tr}\overline{\rho}(\mathcal{M}_i^2) - \operatorname{tr}\overline{\rho}(\mathcal{L}_i^g) \text{ where } i \in \{0,1\}. \tag{5.34}
$$

Curves in $\mathfrak{D}(\mathcal{W})$ corresponding to these curves in the eigenvalue variety can readily be determined. Consider points in $\overline{\mathfrak{E}}_0(\mathcal{W})$ of the form $(1,-1,\overline{u},v)$. The preimage under Φ of such a point is of the form $\varphi^{-1}(w,-w^{-1},w,-w^{-1})$ for some $w \in \mathbb{C}-\{0,\pm 1\}$, and hence $H'(\mathcal{M}_1)=\frac{1+w}{w(1-w)}$ and $H'(\mathcal{L}_1^t)=w^2$.

5.4.6. Eigenvalues at $(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z})=(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$. Using the implicit function theorem as in Subsection 5.3.3, the eigenvalues at the ideal point

$$\xi=(0,1,-1,0,1,-1,0,1,-1,0,1,-1)$$

where $w,x,y,z\to 1$ can be computed. Note that this point maps to a singularity of $\mathfrak{D}'(\mathcal{W})$. We use the projection $(w,x,y,z)\to(w,y,z)$, with image defined by:

$$0=g(w,y,z)=w-wy-wz+yz-wy^2z^2+w^2y^2z^2.$$

The holonomies of meridians take the form

$$H'(\mathcal{M}_0)=-yz\frac{1-w}{1-y},\qquad H'(\mathcal{M}_1)=-\frac{1-z}{yz(1-w)},$$

and as ξ is approached, the eigenvalues of the longitudes converge one. Since the growth rates of all parameters are equal, let

$$w=1+\epsilon,\qquad y=1-\overline{s}^{-1}\epsilon+\epsilon Y\qquad z=1-\overline{u}\epsilon,$$

and we are seeking $Y(\epsilon)$ such that $g(w(\epsilon),y(\epsilon),z(\epsilon))=0$ for all $\epsilon\in B_\delta(0)$ and $Y(0)=0$ for some $\overline{s},\overline{u}\in\mathbb{C}-\{0\}$. As in Subsection 5.3.3, substitution yields a polynomial $F(\epsilon,Y)$, and in this case the implicit function theorem applies if $(\overline{s}-1)(\overline{u}-1)=0$. We have

$$H'(\mathcal{M}_0)=\frac{(1-\overline{u}\epsilon)(1-\overline{s}^{-1}\epsilon+\epsilon Y)}{\overline{s}^{-1}+Y},\qquad H'(\mathcal{M}_1)=\frac{\overline{u}}{(1-\overline{u}\epsilon)(1+\epsilon)},$$

Thus, for any $\overline{s},\overline{u}\in\mathbb{C}-\{0\}$, one can obtain the limiting eigenvalues $(1,1,\overline{u},1)$ and $(\overline{s},1,1,1)$ as one approaches ξ. These eigenvalues are precisely the eigenvalues of the reducible representations in $\overline{\mathfrak{C}}(\mathcal{W})$.

5.4.7. Character varieties. We conclude this chapter with a complete description of the character varieties associated to $\mathcal{W}$. The $SL_2(\mathbb{C})$–Dehn surgery component can be worked out from $\mathfrak{C}(\mathcal{W})$. Let $X = \operatorname{tr}\rho(\mathcal{M}_0), Y = \operatorname{tr}\rho(\mathcal{M}_1)$ and $Z = \operatorname{tr}\rho(\mathcal{M}_0\mathcal{M}_1)$, then $\mathfrak{X}_0(\mathcal{W})$ is defined by the single equation:

$$0 = F(X,Y,Z) = XY + (2 - X^2 - Y^2)Z + XYZ^2 - Z^3.$$

The set of reducible $SL_2(\mathbb{C})$–characters $\mathfrak{X}^r(\mathcal{W})$ is parametrised by $\operatorname{tr}\rho[\mathcal{M}_0, \mathcal{M}_1] = 2$, which is equivalent to:

$$4 = X^2 + Y^2 + Z^2 - XYZ.$$

Thus, $\mathfrak{X}(\mathcal{W}) = \mathfrak{X}_0(\mathcal{W}) \cup \mathfrak{X}^r(\mathcal{W})$.

The form of the relator in (5.10) implies that a $PSL_2(\mathbb{C})$–representation lifts to $SL_2(\mathbb{C})$ if and only if the relator is equal to the identity in $SL_2(\mathbb{C})$ for any assignment of matrices representing the $PSL_2(\mathbb{C})$–representation. Therefore only the Dehn surgery component of the $PSL_2(\mathbb{C})$–character variety can be worked out from the above equation. With $\overline{X} = X^2, \overline{Y} = Y^2$ and $\overline{Z} = Z^2$ one obtains:

$$\begin{aligned} 0 &= \overline{F}(\overline{X},\overline{Y},\overline{Z}) \\ &= -\overline{XY} + (4 - 4\overline{X} + \overline{X}^2 - 4\overline{Y} + \overline{Y}^2)\overline{Z} - (4 - 2\overline{X} - 2\overline{Y} + \overline{XY})\overline{Z}^2 + \overline{Z}^3. \end{aligned}$$

Note that $\overline{F}(X^2, Y^2, Z^2) = F(X,Y,Z)F(-X,Y,Z) = F(X,Y,Z)F(X,-Y,Z)$.

Computation of $PSL_2(\mathbb{C})$–representations which do not lift to $SL_2(\mathbb{C})$ reveals that there is only a finite set, parametrised by $(\overline{X},\overline{Y},\overline{Z}) = (0, 0, 2 \pm \sqrt{2})$. These points do not satisfy $\overline{F}$, and the corresponding representations are irreducible on the peripheral subgroups, with image isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In particular, they do not contribute any points to the $PSL_2(\mathbb{C})$–eigenvalue variety. Thus, all boundary curves which are detected by $\overline{\mathfrak{E}}_0(\mathcal{W})$ are also detected by $\mathfrak{E}_0(\mathcal{W})$.

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