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THE RECIPROCITY OBSTRUCTION FOR RATIONAL POINTS ON COMPACTIFICATIONS OF TORSORS UNDER TORI OVER FIELDS WITH GLOBAL DUALITY

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ABSTRACT. This paper studies the reciprocity obstruction to the local–global principle for compactifications of torsors under tori over a generalised global field of characteristic zero. Under a non-divisibility condition on the second Tate–Shafarevich group for tori, it is shown that the reciprocity obstruction is the only obstruction to the local–global principle. This gives in particular an elegant unified proof of Sansuc's result on the Brauer–Manin obstruction for compactifications of torsors under tori over number fields, and Scheiderer's result on the reciprocity obstruction for compactifications of torsors under tori over p-adic function fields.

Let K be a field of characteristic zero that has $(n+2)$-dimensional global duality in étale cohomology with respect to a collection of n-local fields $K \subset K_v \subset \overline{K}$ indexed by $v \in \Omega_K$. Examples of such fields are totally imaginary number fields (then $n = 1$) and function fields over n-local fields. See Section 1 for details.

Let X be a smooth projective variety over K. Writing $X(\mathbf{A}_K) := \prod_{v\in\Omega} X(K_v)$, we have a *reciprocity pairing*

$$X(\mathbf{A}_K) \times H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(d)) \to \mathbf{Q}/\mathbf{Z}.$$

Writing $X(\mathbf{A}_K)^{\mathrm{rcpr}}$ for the collections of points that pair to zero with every $\omega \in H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(d))$, we have that $X(K) \hookrightarrow X(\mathbf{A}_K)^{\mathrm{rcpr}}$. In particular, when $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ then $X(K) = \emptyset$.

Hence the reciprocity pairing gives an obstruction to the local–global principle. When K is a number field, this obstruction is easily seen to be equivalent to the obstruction coming from the well-known *Brauer–Manin pairing*

$$X(\mathbf{A}_K) \times H^2(X, \mathbf{G}_{\mathrm{m}}) \to \mathbf{Q}/\mathbf{Z}.$$

Main result

In this paper we will show that under a technical assumption on Galois cohomology of tori the reciprocity obstruction is the only obstruction to the local–global principle for any smooth compactification of a torsor under a torus over K (i.e., any smooth projective variety containing a principal homogeneous space under a torus over K as a Zariski-dense open subvariety).

Theorem 1. *Let K be a field of characteristic zero with global duality. Assume that for every torus T over K there is an $N > 0$ such that $Ш^2(K, T)$ is an N-torsion group.*

Then for any smooth compactification X of a torsors under a torus over K we have that $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ if and only if $X(K) = \emptyset$.

Proof. This follows immediately from Corollary 3.3 and Corollary 4.3. □

This generalises (and simplifies the proof of) the original result of Sansuc that for a smooth compactification of a torsor under a torus over a number field the Brauer–Manin obstruction is the only obstruction against the Hasse principle (see [S] and also [Sk]).

The condition on $Ш^2(K,T)$ is not only known to hold for number fields, but also for p-adic function fields (this follows from the duality theorems in [SvH]). In particular, we get a proof of the following unconditional result, due to Scheiderer (private communication), that has not appeared in the literature before.

Corollary. *Let p be a prime and let K be a p-adic function field (i.e., a finite extension of $\mathbf{Q}_p[t]$). Then for any smooth compactification X of a torsor under a torus over K we have that $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ if and only if $X(K) = \emptyset$.*

Method of proof

The proof uses *pseudo-motivic* homology

$$^1H_*(X,\mathbf{Z}) := \mathrm{Ext}^{-*}_{k_{\mathrm{sm}}}(R\Gamma(X/k,\mathbf{G}_{\mathrm{m}}),\mathbf{G}_{\mathrm{m}})$$

as defined in [vH1] for smooth projective varieties over a field of characteristic zero (see Section 2 for some more information).

This homology theory (covariant in X) can be considered to be in between motivic homology and étale homology with coefficients in $\hat{\mathbf{Z}}$ (see [vH1], [vH2]). It is more tractable than motivic homology, but it still contains some important geometric/arithmetic data.

In particular, in certain cases $^1H_0(X,\mathbf{Z})$ can decide whether X has K-rational points.

Theorem 2. *Let X be a smooth compactification of a torsor under a torus over a field k of characteristic zero. Then the degree map*

$$^1H_0(X,\mathbf{Z}) \to {}^1H_0(\mathrm{Spec}\, k,\mathbf{Z}) = \mathbf{Z}$$

is surjective if and only if $X(k) \neq \emptyset$.

Proof. If $X(k) \neq \emptyset$, then functoriality of $^1H_0(-,\mathbf{Z})$ implies the surjectivity of the degree map. The converse follows from Corollary 4.3, since the map $^1H_0(-,\mathbf{Z}) \to {}^1H_0(-,\mathbf{Z})_\tau$ is compatible with the degree map. □

This is the key result in the paper and in fact an easy consequence of Hilbert's Theorem 90 and Steinberg's result that the invertible functions on a torus are characters up to translation. Theorem 1 is then essentially a purely formal consequence of global duality. However, to avoid any unnecessary technical subtleties we will actually derive Theorem 1 from the slightly stronger Corollary 4.3.

As we will see in Section 5, the approach taken here is strongly related to the approach of Colliot-Thélène and Sansuc in the case of number fields: Corollary 4.3 is equivalent to their result that a smooth compactification of a torsor under a torus has rational points if and only if the so-called *elementary obstruction* vanishes. However, the proofs in the present paper are simpler, and extend easily to higher cohomological dimension. This can be explained by the fact that for the varieties under consideration the homological formalism of pseudo-motivic homology happens to be more natural than the dual *co*homological formalism of descent.

Structure of the paper

Most of this paper is devoted to setting up the conceptual framework and establishing its formal properties. In Section 1 we recall the concept of an n-local field, originally due to Parshin, and we will introduce a cohomological global analogue: $(n+2)$-dimensional global duality in étale cohomology. We will introduce the reciprocity pairing in this framework and establish some basic properties. In Section 2 we will recall the definition and basic properties of pseudo-motivic homology. In Section 3 we define a cap-product between pseudo-motivic homology and étale cohomology and we establish a partial duality.

After setting up the proper framework in the first three sections, we show in Section 4 that a principal homogeneous space under a torus actually coincides with the degree 1 part of its zero-dimensional homology. This is essentially a rephrasing of Steinberg's result on the invertible functions on a torus. The main results then follow immediately.

Finally, in Section 5 we will compare the methods used here to other methods in the literature.

1. Higher dimensional local and global duality

1.1. Higher dimensional local duality

In this paper, an *n-local field* (for $n \geq 1$) will be a field k that admits a sequence of fields

$$k_0, k_1, \ldots, k_n = k$$

such that:

- k_0 is a finite field
- For each $i > 0$ the field k_i is the quotient field of an excellent henselian discrete valuation ring $\mathcal{O}_{k_i}$ with residue field k_{i-1}.

A *generalised n-local field* will be a field satisfying the same hypotheses, except that k_0 is only required to be *quasi-finite*, i.e., a perfect field with absolute Galois group isomorphic to $\hat{\mathbf{Z}}$.

A generalised n-local field k with k_1 of characteristic zero satisfies *n-dimensional local duality in étale cohomology*:

- There is a canonical isomorphism $H^{n+1}_{\text{ét}}(k, \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$
- For any finite $\mathrm{Gal}(\bar{k}/k)$-module M and any $i \in \mathbf{Z}$ the Yoneda pairing

  $$H^i_{\text{ét}}(k, M) \times \mathrm{Ext}^{N+1-i}_{\text{ét}}(M, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+1}_{\text{ét}}(k, \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

  is a perfect pairing of finite groups.
- When M is a finite unramified $\mathrm{Gal}(\bar{k}/k)$-module (whose torsion is prime to the characteristic of k_0 if $n = 1$), then the unramified cohomology of M is precisely the annihilator of the unramified cohomology of $\mathcal{H}om(M, \mathbf{Q}/\mathbf{Z}(n))$ in the duality pairing.

For $n = 1$ this is classical (see for example [M, §I.2]), for finite k_0 this is [DW, Th. 1.1, Prop. 1.2]. The proofs easily generalise to the case of quasi-finite k_0.

1.2. **Higher dimensional global duality**

Let K be a field of characteristic zero and suppose we have

- A collection Ω of discrete valuations $v\colon\ K \to \mathbf{Z}$.
- An $n \geq 1$ such that for every $v \in \Omega$ the quotient field K_v^{h} of the henselisation $\mathscr{O}_v^{\mathrm{h}}$ of the discrete valuation ring $\mathscr{O}_v := \{x \in K\colon\ v(k) \geq 0\}$ is an n-local field.
- A ring $\mathscr{O}_K \subset K$ such that for all but finitely many $v \in \mathscr{O}$ we have that $\mathscr{O}_K \subset \mathscr{O}_v$.

We will use the notation $\mathbf{A}_K$ (or simply $\mathbf{A}$) for the ring of *adèles* corresponding to (K,Ω), i.e., the subring of $\prod_{v\in\Omega} K_v^{\mathrm{h}}$ consisting of the $\{x_v\}_{v\in\Omega}$ with $x_v \in \mathscr{O}_v$ for all but finitely many $v \in \Omega$. Since for every finite $\mathrm{Gal}(\bar{k}/k)$-module M we have that M extends to an étale sheaf over an affine open subscheme $U \subset \operatorname{Spec}\mathscr{O}_K$, we may define the *adèlic étale cohomology group*

$$H^*_{\text{ét}}(\mathbf{A}_K,M) := \varinjlim_{\substack{V\subset U\\ \text{open affine}}} \left(\prod_{\substack{v\in\Omega\\ v\in V}} H^*_{\text{ét}}(\mathscr{O}_v^{\mathrm{h}},M) \times \prod_{\substack{v\in\Omega\\ v\notin V}} H^*_{\text{ét}}(K_v^{\mathrm{h}},M) \right).$$

By abuse of notation we write $v \in \operatorname{Spec}\mathscr{O}_K$ if $\mathscr{O}_K \subset \mathscr{O}_v$ and similarly for every affine open subscheme $U \subset \operatorname{Spec}\mathscr{O}_K$. We write $\text{Ш}^i(K,M)$ for the kernel of the map

$$H^i_{\text{ét}}(K,M) \to H^i_{\text{ét}}(\mathbf{A},M).$$

Similarly we define the complex of abelian groups $R\Gamma_{\text{ét}}(\mathbf{A},M)$ for any étale sheaf (or complex of étale sheaves) M over some open subscheme $U \subset \operatorname{Spec}\mathscr{O}_K$. We have a map

$$R\Gamma_{\text{ét}}(K,M) \to R\Gamma_{\text{ét}}(\mathbf{A},M)$$

and we define the complex $R\Gamma(K,\mathbf{A};M)$ to be the complex of abelian groups that makes a triangle

$$R\Gamma_{\text{ét}}(K,\mathbf{A};M) \to R\Gamma_{\text{ét}}(K,M) \to R\Gamma_{\text{ét}}(\mathbf{A},M).$$

As the notation indicates, the corresponding cohomology groups $H^i_{\text{ét}}(K,\mathbf{A};M) := H^i(R\Gamma(K,\mathbf{A};M))$ should be thought of as relative cohomology groups. By definition we have a long exact sequence

$$\cdots \to H^i_{\text{ét}}(K,\mathbf{A};M) \to H^i_{\text{ét}}(K,M) \to H^i_{\text{ét}}(\mathbf{A},M) \to H^{i+1}_{\text{ét}}(K,\mathbf{A};M) \to \ldots$$

Remark 1.1. The relative cohomology groups $H^*_{\text{ét}}(K,\mathbf{A};-)$ can be thought of as the cohomology with compact supports of $\operatorname{Spec} K$ regarded as something very open in a compactification of $\operatorname{Spec}\mathscr{O}_K$ (compare [M, §II.2]). This way of seeing it is more in line with Grothendieck–Verdier approach to cohomology and duality. However, a notation H^*_{c} can lead to confusion when studying the cohomology of varieties over K, so the 'Eilenberg–MacLane'-style of notation as relative cohomology seems more convenient.

For any finite $\mathrm{Gal}(\overline{K}/K)$-module M, any $i,\ j \in \mathbf{Z}$ we have that an $\omega \in \mathrm{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j))$ induces maps

$$H^q_{\text{ét}}(K, \mathbf{A}; M) \to H^{q+i}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$$
$$H^q_{\text{ét}}(K, M) \to H^{q+i}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(j))$$
$$H^q_{\text{ét}}(\mathbf{A}, M) \to H^{q+i}_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$$

which are compatible with the long exact sequences of the pair $(K, \mathbf{A})$. Allowing ω to vary we get the *Yoneda pairings*

$$H^q_{\text{ét}}(K, \mathbf{A}; M) \otimes \mathrm{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)),$$
$$H^q_{\text{ét}}(K, M) \otimes \mathrm{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(j)),$$
$$H^q_{\text{ét}}(\mathbf{A}, M) \otimes \mathrm{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)).$$

We say that K has $(n+2)$-dimensional global duality in étale cohomology if:

- We have an isomorphism $H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$ such that the boundary map $H^{n+1}_{\text{ét}}(\mathbf{A}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))$ corresponds to the summation map $\bigoplus_{v \in \Omega} \mathbf{Q}/\mathbf{Z} \xrightarrow{\Sigma} \mathbf{Q}/\mathbf{Z}$.
- For every finite $\mathrm{Gal}(\overline{K}/K)$-module M and any $i \in \mathbf{Z}$ the Yoneda pairing
$$H^i_{\text{ét}}(K, \mathbf{A}; M) \times \mathrm{Ext}^{n+2-i}_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$$
is a perfect pairing of abelian groups.

As a purely formal consequence we get duality for any bounded complex $\mathscr{C}$ of constructible étale sheaves defined over an open subset $U \subset \operatorname{Spec} \mathscr{O}_K$: we have that

$$H^i_{\text{ét}}(K, \mathbf{A}; \mathscr{C}) \times \mathrm{Ext}^{n+2-i}_{K_{\text{ét}}}(\mathscr{C}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$$

is a perfect pairing of abelian groups for every $i \in \mathbf{Z}$.

Examples of fields that satisfy $(n+2)$-dimensional global duality in étale cohomology are

- Totally imaginary number fields (with $n = 1$)
- Function fields of curves over generalised $(n-1)$-local fields with k_1 of characteristic zero.

Remark 1.2. To get 3-dimensional global duality for number fields that admit real embeddings, one needs to take care of the real places separately (as in [M, §II.2]). Having done that, the methods of this paper still apply.

1.3. **The reciprocity pairing**

Let X be a smooth projective variety over a field K having $(n+2)$-dimensional global duality in étale cohomology.

For any $i,\ j \in \mathbf{Z}$ the restriction map gives pairings of sets

$$X(K) \times H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(K; \mathbf{Q}/\mathbf{Z}(j))$$
$$X(\mathbf{A}) \times H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)).$$

Here

$$H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j)) := H^i_{\text{ét}}(\mathbf{A}, R\Gamma(X/K, \mathbf{Q}/\mathbf{Z}(j))).$$

When we compare these two pairings, we see that composition with the restriction map $H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j))$ and the boundary map

$H^i_{\text{ét}}(\mathbf{A};\mathbf{Q}/\mathbf{Z}(j)) \to H^{i+1}_{\text{ét}}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(j))$ transforms the second pairing into a pairing

$$X(\mathbf{A}) \times H^i_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(j)) \to H^{i+1}_{\text{ét}}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(j))$$

with the property that the image of the map $X(K) \to X(\mathbf{A})$ lands into the subset

$$X(\mathbf{A})^{\perp H^i_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(j))} := \left\{\{x_v\} \in X(\mathbf{A}) \colon \langle\{x_v\},\omega\rangle = 0 \text{ for any } \omega \in H^i_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(j))\right\}$$

Taking $i = n+1$, $j = n$, we get the *reciprocity pairing*

$$X(\mathbf{A}) \times H^{n+1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(j)) = \mathbf{Q}/\mathbf{Z}$$

mentioned in the introduction and the fact that

$$X(K) \hookrightarrow X(\mathbf{A})^{\text{rcpr}} = X(\mathbf{A})^{\perp H^{n+1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(n))}.$$

1.4. **Generalised global duality beyond finite coefficients**

Later in this paper we will use $(n+2)$-dimensional global duality to detect elements in $H^2_{\text{ét}}(K,\mathbf{A};\mathrm{X}(M))$ for a finitely generated group scheme over K. Here a *finitely generated group scheme* over a perfect field k is a group scheme G such that $G(\bar{k})$ is a finitely generated group. and

$$\mathrm{X}(M) := \mathscr{H}om(M,\mathbf{G}_{\text{m}})$$

is the Cartier dual of M.

Morally speaking, one would expect a perfect pairing

$$H^n_{\text{ét}}(K,M \overset{L}{\otimes} \mathbf{Z}(n-1))) \times H^2_{\text{ét}}(K,\mathbf{A};\mathrm{X}(M)) \to H^{n+3}_{\text{ét}}(K,\mathbf{A};\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z}$$

for suitable 'motivic' complexes of sheaves $\mathbf{Z}(n-1)$ and $\mathbf{Z}(n)$ (recall that in the motivic formalism we have $\mathbf{G}_{\text{m}} = \mathbf{Z}(1)[1]$). This holds indeed for K a number field (cf. [M, §I.4]), but I do not know of such a full duality in any other case (nevertheless, for K a p-adic function field, the results of [SvH] give the required duality between $H^2_{\text{ét}}(K,M \otimes^L \mathbf{Z}(1))$ and $H^2_{\text{ét}}(K,\mathbf{A};\mathrm{X}(M))$ without introducing a complex $\mathbf{Z}(2)$). Besides, to apply such a result to a reciprocity pairing on a projective variety X, one would not just need $\mathrm{Pic}_{X/K}$ to be finitely generated in order to play the role of M, but one would also have to prove that in that case $\mathscr{H}^2_{\text{ét}}(X/K,\mathbf{Z}(n)) = \mathrm{Pic}_{X/K} \otimes \mathbf{Z}(n-1)$.

To avoid these complications, we look at the Yoneda pairing

(1)
$$H^{n-1}_{\text{ét}}(K,M \overset{L}{\otimes} \mathbf{Q}/\mathbf{Z}(n-1))) \times H^2_{\text{ét}}(K,\mathbf{A};\mathrm{X}(M)) \to H^{n+2}_{\text{ét}}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z}$$

associated to the isomorphisms

$$M \overset{L}{\otimes} \mathbf{Z}/m(n-1) \simeq R\mathscr{H}om(\mathrm{X}(M),\mathbf{G}_{\text{m}} \overset{L}{\otimes} \mathbf{Z}/m(n-1)) = R\mathscr{H}om(\mathrm{X}(M),\mathbf{Z}/m(n)[1])$$

for all $m \in \mathbf{N}$.

Proposition 1.3. *Let K be a field that has $(n+2)$-dimensional global duality, and let M be a finitely generated group scheme over K. If $\text{Ш}^2(K,\mathrm{X}(M))$ is purely N-torsion for some $N \in \mathbf{N}$, then the pairing* (1) *is nondegenerate on the right.*

Proof. By Hilbert's Theorem 90 and a trace argument we have that there is an $N' \in \mathbf{N}$ such that $H^1_{\text{ét}}(\mathbf{A},\mathrm{X}(M))$ is purely N'-torsion. Hence the long exact sequence of relative cohomology and the hypothesis on Ш^2 implies that $H^2_{\text{ét}}(K,\mathbf{A};\mathrm{X}(M))$ embeds into $H^2_{\text{ét}}(K,\mathbf{A};\mathrm{X}(M) \otimes^L \mathbf{Z}/NN')$.

Global duality then implies that $H^2_{\text{ét}}(K,\mathbf{A};\mathrm{X}(M))$ embeds into the dual of $H^{n-1}_{\text{ét}}(K, M\otimes^L \mathbf{Z}/NN'(n-1)))$, hence into the dual of $H^{n-1}_{\text{ét}}(K, M\otimes^L \mathbf{Q}/\mathbf{Z}(n-1)))$. □

We will also use the following easy lemma.

Lemma 1.4. *Let K be a field that has $(n+2)$-dimensional global duality. The pairing*

$$H^{n+1}_{\acute{e}t}(K,\mathbf{Q}/\mathbf{Z}(n))\times H^1_{\acute{e}t}(K,\mathbf{A};\mathbf{Z})\to H^{n+2}_{\acute{e}t}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(n)))=\mathbf{Q}/\mathbf{Z} \tag{2}$$

is nondegenerate on the right.

Proof. This follows easily from the fact that $H^1_{\text{ét}}(K,\mathbf{A};\mathbf{Z}) = (\prod_{v\in\Omega}\mathbf{Z})/\mathbf{Z}$, whereas $H^{n+1}_{\text{ét}}(K,\mathbf{Q}/\mathbf{Z}(n))$ surjects onto the kernel of the map $\bigoplus_{v\in\Omega}\mathbf{Q}/\mathbf{Z} = H^{n+1}_{\text{ét}}(\mathbf{A},\mathbf{Q}/\mathbf{Z}(n))\to H^{n+2}_{\text{ét}}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(n)))=\mathbf{Q}/\mathbf{Z}$. □

2. Pseudo-motivic homology

Let k be a field of characteristic zero. Let X be a nonsingular variety over k. We write X_{sm} for the smooth site over X (i.e., underlying category the smooth schemes of finite type over X and coverings the surjective smooth morphisms).

For any sheaf $\mathscr{F}$ on X_{sm} we denote by $R\Gamma(X/K_{\text{sm}},\mathscr{F})$ the total direct image in the derived category of sheaves on $(\operatorname{Spec} k)_{\text{sm}}$ of $\mathscr{F}$ under the structure morphism $X\to\operatorname{Spec} k$.

We put

$$\begin{aligned}
\mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}}) &:= R\Gamma(X/k_{\mathrm{sm}},\mathbf{G}_{\mathrm{m}})\\
\mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}})_\tau &:= \tau_{\le 1}R\Gamma(X/k_{\mathrm{sm}},\mathbf{G}_{\mathrm{m}})\\
\mathscr{C}^*(X,\mathbf{Z}/m(j)) &:= R\Gamma(X/k_{\mathrm{sm}},\mathbf{Z}/m(j))\\
\mathscr{C}^*(X,\mathbf{Z}/m(j))_\tau &:= \mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}})_\tau\overset{L}{\otimes}\mathbf{Z}/m(j-1)\\
\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z}) &:= R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}}),\mathbf{G}_{\mathrm{m}})\\
\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z})_\tau &:= R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}})_\tau,\mathbf{G}_{\mathrm{m}})\\
\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z}/m)_{(\tau)} &:= R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathscr{C}^*(X,\mathbf{Z}/m)_{(\tau)},\mathbf{Z}/m)\\
H^i(X,-)_{(\tau)} &:= H^i(k_{\mathrm{sm}},\mathscr{C}^*(X,-)_{(\tau)})\\
{}^1H^{\mathrm{c}}_i(X,\mathbf{Z})_{(\tau)} &:= H^{-i}(k_{\mathrm{sm}},\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z})_{(\tau)})\\
H^{\mathrm{c}}_i(X,\mathbf{Z}/m)_{(\tau)} &:= H^{-i}(k_{\mathrm{sm}},\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z}/m)_{(\tau)})\\
{}^1\mathscr{H}^{\mathrm{c}}_i(X,\mathbf{Z}/m) &:= \mathscr{H}^{-i}(\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z}/m)_{(\tau)})\ \text{ for } i=0,\,1
\end{aligned}$$

Here ${}^1H^{\mathrm{c}}_i(X,\mathbf{Z})$ is *pseudo-motivic homology with compact support*, and $H^{\mathrm{c}}_i(X,\mathbf{Z}/m)$ is ordinary étale homology with compact supports. The two are related by a Kummer exact sequence. When X is projective, there is no need to specify that we use compact supports, and we will just write ${}^1H_i(X,\mathbf{Z})$.

For technical reasons we will work with the truncated version ${}^1H^{\mathrm{c}}_i(X,\mathbf{Z})_\tau$, which is somewhat easier to work with. By definition, it fits with $H^{\mathrm{c}}_i(X,\mathbf{Z}/m)_\tau$ into a Kummer exact sequence.

Remark 2.1. We only need the smooth topology in the definition of the complexes $\mathscr{C}_*^{\mathrm{c}}(X,\mathbf{Z})_{(\tau)}$. After that, the comparison between smooth cohomology and étale cohomology assures that we might as well compute everything on the étale site, and get the same results In particular, $H^i(k_{\mathrm{sm}},\mathscr{C}^*(X,\mathbf{Z}/m(j))) = H^i_{\mathrm{\acute{e}t}}(X,\mathbf{Z}/m(j))$.

2.1. **Some calculations**

In the present paper we are interested in varieties with a finitely generated Picard scheme. For these varieties the truncated pseudo-motivic homology has a very simple structure.

Lemma 2.2. *Assume X is a smooth projective geometrically irreducible variety over k such that the Picard scheme $\mathrm{Pic}_{X/k}$ is a finitely generated group scheme. Then we have a triangle*

$$\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}})[1] \to \mathscr{C}_*(X,\mathbf{Z})_\tau \to \mathbf{Z}.$$

Proof. By Cartier duality this follows from the fact that we have a triangle

$$\mathbf{G}_{\mathrm{m}} \to \mathscr{C}(X,\mathbf{G}_{\mathrm{m}})_\tau \to \mathrm{Pic}_{X/K}[-1].$$

□

Corollary 2.3. *With X as above, we have a long exact sequence*

$$\cdots \to H^1(k,\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}})) \to {}^1H_0(X,\mathbf{Z})_\tau \to H^0(k,\mathbf{Z}) \\ \to H^2(k,\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}})) \to \cdots$$

Lemma 2.4. *Assume V is a smooth geometrically connected variety over k such that $\mathrm{Pic}_{V/k}=0$. Then we have a triangle*

$$\mathscr{H}om(\bar{k}[V]^*/\bar{k}^*,\mathbf{G}_{\mathrm{m}}) \to \mathscr{C}_*(V,\mathbf{Z})_\tau \to \mathbf{Z}$$

Proof. It is not hard to deduce from the discussion in [vH2, Sec. 1.2] that we have a triangle

$$\mathbf{G}_{\mathrm{m}} \to \mathscr{C}^*(V,\mathbf{Z})_\tau \to M.$$

where M is (the complex concentrated in degree zero associated to the sheaf represented by) a finitely generated group scheme over k. Checking the global sections over $\bar{k}$ then gives that M is the group scheme corresponding to the finitely generated Galois module $\bar{k}[V]^*/\bar{k}^*$. □

Lemma 2.5. *Let X be a nonsingular projective variety over k and let $V\subset X$ be an open subvariety, then the natural map*

$${}^1H_0^{\mathrm{c}}(V,\mathbf{Z})_\tau \to {}^1H_0(X,\mathbf{Z})_\tau$$

is surjective

Proof. This is part of [vH2, Cor. 1.5]. □

2.2. **Homology classes of points**

For any variety V over k we have that the covariantly functorial properties of pseudo-motivic homology give a natural map

$$V(k) \to {}^1H_0^{\mathrm{c}}(V,\mathbf{Z}).$$

We denote the homology class of a k-valued point $x \in V(k)$ by $[x]$, If x corresponds to a map $i\colon \operatorname{Spec} k \to V$ then $[x]$ corresponds to the morphism

$$R\Gamma(X/k, \mathbf{G}_{\mathrm{m}}) \to \mathbf{G}_{\mathrm{m}}$$

induced by the natural morphism

$$\mathbf{G}_{\mathrm{m}} \to i_* \mathbf{G}_{\mathrm{m}}$$

of sheaves on X. We will not make a distinction in notation between the class $[x] \in {}^1H_0^{\mathrm{c}}(V, \mathbf{Z})$ and its image under the truncation map ${}^1H_0^{\mathrm{c}}(V, \mathbf{Z}) \to {}^1H_0^{\mathrm{c}}(V, \mathbf{Z})_\tau$

The sheafified version of this map gives a morphism of sheaves (of sets) over k_{sm}

$$V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z})$$

with the image of V landing in the inverse image of 1 under the degree map

$${}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) \to \mathbf{Z}.$$

See [vH1] and [vH2] for more information.

Lemma 2.6. *Assume V is a smooth geometrically connected variety over k such that* $\operatorname{Pic}_{V/k} = 0$. *Then the morphism*

$$V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) = R\mathscr{H}om_{k_{sm}}(\Gamma(V/k_{sm}, \mathbf{G}_{\mathrm{m}}), \mathbf{G}_{\mathrm{m}})$$

is given by locally sending a section $x \in V$ to the map that sends a local section f of $\Gamma(V/k_{sm}, \mathbf{G}_{\mathrm{m}})$ to $f(x)$.

Proof. This follows immediately from the definitions. □

3. The cap-product and partial generalised global duality for pseudo-motivic homology

3.1. Definition and basic properties of the cap product

Let X be a smooth variety over a field k of characteristic zero. Since $\mathscr{C}_*^{\mathrm{c}}(X, \mathbf{Z}) = R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathscr{C}^*(C, \mathbf{G}_{\mathrm{m}}), \mathbf{G}_{\mathrm{m}})$, we have well-defined Yoneda-products

$${}^1H_i^{\mathrm{c}}(X, \mathbf{Z}) \times H^j(X, \mathbf{G}_{\mathrm{m}}) \to H^{j-i}(K, \mathbf{G}_{\mathrm{m}})$$
$${}^1H_i^{\mathrm{c}}(X, \mathbf{Z}) \times H^j(X, \mathbf{Q}/\mathbf{Z}(1)) \to H^{j-i}(K, \mathbf{Q}/\mathbf{Z}(1)).$$

Applying Tate twist to the torsion coefficients in the second pairing gives us

$${}^1H_i^{\mathrm{c}}(X, \mathbf{Z}) \times H^j(X, \mathbf{Q}/\mathbf{Z}(m)) \to H^{j-i}(K, \mathbf{Q}/\mathbf{Z}(m)).$$

for any $m \in \mathbf{Z}$. Similarly, we have the truncated versions

$${}^1H_i^{\mathrm{c}}(X, \mathbf{Z})_\tau \times H^j(X, \mathbf{G}_{\mathrm{m}})_\tau \to H^{j-i}(K, \mathbf{G}_{\mathrm{m}})$$
$${}^1H_i^{\mathrm{c}}(X, \mathbf{Z})_\tau \times H^j(X, \mathbf{Q}/\mathbf{Z}(m))_\tau \to H^{j-i}(K, \mathbf{Q}/\mathbf{Z}(m)).$$

All these pairings can be called *cap-product pairings* and will be denoted by $- \cap -$.

For a k-valued point $x\colon \operatorname{Spec} k \hookrightarrow X$, and an $\omega \in H^j(X, \mathbf{Q}/\mathbf{Z}(m))_{(\tau)}$ we have that

$$[x] \cap \omega = i^* \omega \in H^j(k, \mathbf{Q}/\mathbf{Z}(m)). \tag{3}$$

This follows easily from the definitions, in particular from the fact that the homology class $[x]$ is defined using the the natural maps $\mathbf{G}_{\mathrm{m}} \to i_*\mathbf{G}_{\mathrm{m}}$ and the pull-back homomorphism i^* is defined using the natural map $\mathbf{Q}/\mathbf{Z}(m) \to i_*\mathbf{Q}/\mathbf{Z}(m)$.

When K is a field with global $(n+2)$-dimensional duality in étale cohomology, we also get

$$^1H_i^{\mathrm{c}}(X_{\mathbf{A}},\mathbf{Z})_{(\tau)} \times H^j(X,G)_{(\tau)} \to H^{j-i+1}(K,\mathbf{A};G)$$
$$^1H_i^{\mathrm{c}}(X,X_{\mathbf{A}};\mathbf{Z})_{(\tau)} \times H^j(X,G)_{(\tau)} \to H^{j-i+1}(K,\mathbf{A};G).$$

for $G = \mathbf{G}_{\mathrm{m}}$ or $\mathbf{Q}/\mathbf{Z}(m)$, and equation (3) gives a commutative diagram of pairings

$$\begin{array}{ccccc} V(\mathbf{A}) & \times & H^{n+1}(V,\mathbf{Q}/\mathbf{Z}(m) & \xrightarrow{\text{rcpr}} & \mathbf{Q}/\mathbf{Z} \\ \downarrow & & \uparrow & & \| \\ ^1H_0^{\mathrm{c}}(V_{\mathbf{A}},\mathbf{Z})_{(\tau)} & \times & H^{n+1}(V,\mathbf{Q}/\mathbf{Z}(m)) & \xrightarrow{\cap} & \mathbf{Q}/\mathbf{Z} \end{array} \tag{4}$$

3.2. **Partial generalised global duality for pseudo-motivic homology**

Theorem 3.1. *Let X be a smooth projective variety over a field K having $(n+2)$-dimensional global duality in étale cohomology. Assume that* $\mathrm{Pic}_{X/K}$ *is a finitely generated* $\mathrm{Gal}(\overline{K}/K)$*-module and that* $Ш^2(K,\mathrm{X}(\mathrm{Pic}_{X/K}))$ *is purely N-torsion for some $N \in \mathbf{N}$. Then the cap-product pairing*

$$^1H_{-1}(X,X_{\mathbf{A}};\mathbf{Z})_\tau \times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to H^{n+2}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

is nondegenerate on the left.

Proof. The triangle of Lemma 2.2 gives the following diagram of compatible pairings with exact rows

$$\begin{array}{ccccc} 0 \longrightarrow H^2(K,\mathbf{A};\mathrm{X}(\mathrm{Pic}_{X/K})) & \longrightarrow & ^1H_{-1}(X,X_{\mathbf{A}};\mathbf{Z})_\tau & \longrightarrow & H^1(K,\mathbf{A};\mathbf{Z}) \\ \times & & \times & & \times \\ 0 \leftarrow H^{n-1}(K,\mathrm{Pic}_{X/K}\otimes^L \mathbf{Q}/\mathbf{Z}(n-1)) & \leftarrow & H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau & \leftarrow & H^{n+1}(K,\mathbf{Q}/\mathbf{Z}(n)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Q}/\mathbf{Z} & = & \mathbf{Q}/\mathbf{Z} & = & \mathbf{Q}/\mathbf{Z} \end{array}$$

where the second pairing is the pairing (1) and the fourth pairing is the pairing of Lemma 1.4. Since both pairings are nondegenerate on the $(K,\mathbf{A})$-side, it follows that the third pairing is nondegenerate on the $(K,\mathbf{A})$-side as well. □

Corollary 3.2. *Let X be as in Theorem 3.1. Then the left kernel of the pairing*

$$^1H_0(X_{\mathbf{A}};\mathbf{Z})_\tau \times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}$$

is precisely the image of the map

$$^1H_0(X,\mathbf{Z})_\tau \to H_0(X_{\mathbf{A}},\mathbf{Z})_\tau$$

Proof. This follows from Theorem 3.1, the exact sequence for the cohomology of the pair $(K,\mathbf{A})$, and the fact that we have a compatible diagram of pairings

$$\begin{array}{ccccc} ^1H_0(X_{\mathbf{A}};\mathbf{Z})_\tau & \times & H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau & \to & \mathbf{Q}/\mathbf{Z} \\ \downarrow f & & \| & & \| \\ ^1H_{-1}(X,X_{\mathbf{A}};\mathbf{Z})_\tau & \times & H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau & \to & \mathbf{Q}/\mathbf{Z} \end{array}$$

□

Corollary 3.3. *Let X be as in Theorem 3.1. If $X(\mathbf{A})^{\mathrm{rcpr}} \neq \emptyset$, then the degree map*

$$H_0(X,\mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective.

Proof. Take an adèlic point $\{x_v\} \in X(\mathbf{A})^{\mathrm{rcpr}}$. The compatibility between cap-product and the map $X(\mathbf{A}) \to {}^1H_0(X_{\mathbf{A}},\mathbf{Z})_\tau$ implies that its homology class

$$[\{x_v\}] \in {}^1H_0(X_{\mathbf{A}},\mathbf{Z})_\tau$$

is orthogonal to any $\omega \in H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))$, so certainly to any $\omega \in H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau$. Therefore, the homology class $[\{x_v\}]$ is the restriction of some $\gamma \in {}^1H_0(X,\mathbf{Z})$. Since each $[x_v] \in {}^1H_0(X_{K_v},\mathbf{Z})$ is of degree 1, the degree of γ is 1. $\square$

4. **Pseudo-motivic homology of toric varieties**

Proposition 4.1. *Let V be a torsor under a torus T over k.*

(i) *The triangle of 2.4 is naturally isomorphic to the triangle*

$$T \to {}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z})_\tau \to \mathbf{Z}$$

(ii) *The natural map $V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z})_\tau$ induces a T-equivariant isomorphism of V with the connected component of ${}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z})_\tau$ mapping to $1 \in \mathbf{Z}$.*

Proof. The first part of the proposition follows by Cartier duality from the sheafified version of Rosenlicht's result that we have a short exact sequence

$$0 \to \mathbf{G}_{\mathrm{m}} \to \Gamma(V/k_{\mathrm{sm}},\mathbf{G}_{\mathrm{m}}) \to \mathrm{X}(T) \to 0.$$

See [Ro], and also [Ray, Cor. VII.1.2.], [CTS, Prop. 1.4.2].

To get the second part of the proposition, we need the extra information that the map $\Gamma(V/k_{\mathrm{sm}},\mathbf{G}_{\mathrm{m}}) \to \mathrm{X}(T)$ considered above is defined locally by sending a local section f of $\Gamma(V/k_{\mathrm{sm}},\mathbf{G}_{\mathrm{m}})$ to the map that sends a local section t of T to $f(t\cdot x)/f(x)$ for any local section x of V. Comparing this with the description of the map $V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z})_\tau$ in Lemma 2.6 gives the desired result. $\square$

Corollary 4.2. *Let V be a torsor under a torus T over k. For any field extension k'/k we have that the natural map*

$$V(k') \to {}^1H_0(V_{k'},\mathbf{Z})_\tau$$

gives a $T(k')$-equivariant isomorphism of $V(k')$ onto the subset of elements of ${}^1H_0^{\mathrm{c}}(V_{k'},\mathbf{Z})_\tau$ of degree 1.

Corollary 4.3. *Let V be a torsor under a torus T over k. Let X be a smooth projective variety over k containing V as a Zariski-dense subvariety.*

(i) *The degree map*

$${}^1H_0^{\mathrm{c}}(V,\mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective if and only if $V(k') \neq \emptyset$.

(ii) *The degree map*

$${}^1H_0(X,\mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective if and only if $X(k) \neq \emptyset$.

Proof. The first statement follows immediately from Corollary 4.2, whereas the second statement follows from the first combined with Lemma 2.5. $\square$

Remark 4.4. It is clear from the above, that we can sharpen Theorem 1 by replacing the full group $H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))$ in the reciprocity pairing by the truncated group $H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau$, or its image under the map $H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))$.

In the case of a number field, this makes no difference, since the so-called 'algebraic' cohomological Brauer group, (which is $H^2(X,\mathbf{G}_\mathrm{m})_\tau$ in our notation) is equal to the full cohomological Brauer group $H^2(X,\mathbf{G}_\mathrm{m})$.

5. Comparison with the literature

Torsors under a torus T over a (generalised) global field K which are trivial everywhere locally are classified by $\text{Ш}^1(K,T)$. It follows from Rosenlicht's result and Hilbert Theorem 90 that $H^1(K,T)$ embeds into $H^2(K,\mathrm{X}(\mathrm{Pic}_{X/K}))$ for any smooth compactification X of a principal homogeneous space V under T. Under the assumptions of Theorem 1, duality then gives that $\text{Ш}^1(K,T)$ embeds into the dual of $H^{n-1}(K,\mathrm{Pic}_{X/K}\otimes^L\mathbf{Q}/\mathbf{Z}(n-1))$, hence into the dual of $H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau$. Therefore, it is not very surprising that the reciprocity pairing detects any failure of the local–global principle.

The only problem is to relate the abstract 'arithmetic' pairing

$$\text{Ш}^1(K,T)\times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}$$

to the 'geometric' reciprocity pairing

$$X(\mathbf{A})\times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}.$$

We have seen that pseudo-motivic homology provides a nice conceptual intermediate to compare the two pairings, but there have been other approaches as well. The existing literature deals with number fields, so here we consider the Brauer group, rather than $H^2(X,\mathbf{Q}/\mathbf{Z}(1))_\tau$.

In [S] the comparison between the 'geometric' and the 'arithmetic' pairing is essentially done in Lemma 8.4, using explicit ways of representing classes in the Brauer group and explicit cochain calculations. If one would want to apply this approach to global duality fields of higher cohomological dimension, both the higher degree of the cochains and the fact that the coefficients would be in $\mathbf{Q}/\mathbf{Z}(n)$ should complicate things considerably.

A more conceptual approach, due to Colliot-Thélène and Sansuc, and described in [CTS] uses the concept of a *universal X-torsor* under groups of multiplicative type. The most streamlined version of this approach is probably presented in [Sk]. As in the present paper, the proof proceeds in two major steps. The first result is that for any smooth projective variety X over a number field K with $X(\mathbf{A}_K)^{\mathrm{Br}(X)_\tau}\neq\emptyset$ we have that the universal X-torsor exists, and the second result is that for a smooth projective toric variety over K the universal X-torsor exists if and only if $X(K)\neq\emptyset$.

There is a very clear relation with the present paper: Colliot-Thélène and Sansuc show that the universal X-torsor exists if and only if the 2-fold extension of Galois modules

$$0\to\bar{k}^*\to\bar{k}(X)^*\to\mathrm{Div}(\overline{X})\to\mathrm{Pic}(\overline{X})\to 0$$

is trivial. This can be seen to be equivalent to the surjectivity of the degree map

$${}^1H_0(X,\mathbf{Z})_\tau\to\mathbf{Z}.$$

Therefore, the two steps of the proof are equivalent, but in both steps the methods of proof are different. In particular in the first step the homology approach of the present paper seems much more efficient than the approach of Colliot-Thélène and Sansuc (or Skorobogatov's streamlined version in [Sk, Sec. 6.1]), where again the core of the proof is a comparison of the 'geometric' and the 'arithmetic' pairing using cocycle computations.

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