Singularities of Complex Plane Curves

Using blow ups to resolve singularities

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Classifying points of a curve

Definition

1. Take $f \in \mathbb{C}[x, y]$, and let C be the curve defined by $f(x, y)$ in $\mathbb{C}^2$. Then the multiplicity of a point $p \in C$ is the order of the lowest non-vanishing term in the Taylor Series Expansion of f at p, denoted $\nu_p(C)$.
2. If $\nu_p(C) = 1$, we call p a *regular* point of C, and if $\nu_p(C) > 1$, then p is a *singular* point of C.
3. If the number of distinct tangents at $p \in C$ is $\nu_p(C)$, and $\nu_p(C) > 1$, then p is an ordinary multiple point.
4. A curve C is non-singular if every point P of C is regular.

Blowing up $\mathbb{C}^2$

Definition

Let $B = \{\big((x,y),\big[u:v\big]\big)\,|xv = yu\} \subset \mathbb{C}^2 \times \mathbb{CP}^1$, and

$$\pi : B \to \mathbb{C}^2, ((x,y),[u:v]) \mapsto (x,y)$$

Then B is a blow up of $(0,0) \in \mathbb{C}^2$ and we call

$$E := \pi^{-1}((0,0))$$

an exceptional line.

Blowing up a point of a curve

Given a curve $C \subset \mathbb{C}^2$, we call

$$C' := \overline{\pi^{-1}(C \setminus \{(0,0)\})}$$

the strict preimage of C.

Getting another plane curve

We can identify the subset $B \cap (\mathbb{C}^2 \times M_1)$ isomorphically with $\mathbb{C}^2$ through the map

$$\tilde{\pi}((x, xv), [1 : v]) = (x, v)$$

where M_1 is the first coordinate chart on $\mathbb{CP}^1$. Then $\tilde{\pi}(C')$ is again a plane curve, called the strict transform of C.

Standard Resolution

We can iteratively perform blow ups, generating a sequence of strict transforms

$$C_i = \overline{\pi_i^{-1}(C_{i-1} \setminus \{(0,0)\}}$$

and define $E = \cup_{k=1}^{k} E_i$, where E_k is the k^{th} exceptional line.
If for all $j > k$ C_j is non-singular, intersects E transversally, and for each $p' \in C_i \cap E$ there is a unique k such that $p' \in E_k$, then C_k is the standard resolution of C.

Blow ups of $\mathbb{CP}^2$

Definition

Blowing up a point in $\mathbb{CP}^2$ is done by choosing an affine coordinate chart and blowing up the point in $\mathbb{C}^2$.

Any singularity can be resolved

Theorem

Given a curve $C \subset \mathbb{CP}^2$, the singularities can be resolved by a finite sequence of blow ups.

Multiplicity Sequences

Given a sequence of blow ups resolving the singularity p of the curve C, we have a sequence of multiplicites $\{\nu_{p_i}(C_i)\}_{i=0}^{k-1}$ where C_0 is the original curve, and C_k is the standard resolution.