Algebraic Topology, Semester 1, 2015, Zhou Zhang

Weeks 1 to 13

Following Chapters 0, 1 and 2 in "Algebraic Topology"
by Allen Hatcher

Overview

Weeks 1-2: Chapter 0, Useful Geometric Notions

Weeks 2-7: Chapter 1, Fundamental Group

Weeks 7-13: Chapter 2, Homology

Week 13: Wrap-up

Before We Start

The struggle between intuitive idea and rigorous argument is going to be evident along the way. Find your own balance and keep up with the reading.

Chapter 0: Useful Geometric Notions

Homotopy and Homotopy Type

1. Background information.

Same topology (i.e., collections of open sets): homeomorphism (continuous map and inverse).

[This is so-called ”rubber geometry”, i.e. you can stretch and squeeze the space.]

Sometimes (actually very frequently), it contains too much information (or asks for too much detail) for practical goals and we want more compact information (to tell things apart, for example, i.e., classification problem).

One of those things is our main topic here, ”homotopy” or ”homotopy type”.

[Intuitively, we use ideal rubber and would be able to squeeze and stretch much harder and more creatively than what children can do to their soft toys.]

Conventions:

- The maps are all continuous: we are talking about ”topology”, which cares and only cares about continuous structure.
- $I = [0, 1]$.

2. Special case (of homotopy).

Example: annulus shrinks to circle.

More generally, **deformation retraction** as follows.

A deformation retraction of X to a subspace A is the following package of information:

$$f : X \times I \to X \text{ such that}$$

i) $f_0(x) := f(x, 0) = x$ and $f_1 := f(x, 1) \in A$ for $x \in X$;
ii) $f(a, t) = a$ for $a \in A$ and $t \in I$.
We call A is a deformation retract of X.
(Recall the topology of $X \times I$. Equivalently, we say $f_t : X \to X$ is a continuous family of (continuous) maps.)

In the above example, X is annulus, A is circle, and in polar coordinates for $r \in [1/2, 3/2]$, the homotopy map is

$$f(r, \theta, t) = (r^{1-t}, \theta)$$

which is certainly not unique.

The deformation retract A of X is by no means unique (example: different type of glass frames for a two-hole disk. (Hatcher, Page 2)) but not too different [”same homotopy type”].

3. General notions.

a) Homotopy (between maps).

$f, g : X \to Y$ are **homotopic** if there exists $F : X \times I \to Y$ with $F_0 = f$ and $F_1 = g$, denoted as $f \simeq g$. (All continuous maps!)

$f, g : X \to X$ are **homotopic relative to a subspace A** if there exists $F : X \times I \to X$ with $F_0 = f$, $F_1 = g$ and $F(a, t) = a$ for $a \in A$ and $t \in I$, denoted as $f \simeq_A g$.

Claim: they are equivalence relations.

b) Homotopy equivalence (between topology spaces)

X and Y are homotopy equivalent (or of the same homotopy type) if there exists $f : X \to Y$ with a homotopy "inverse" $g : Y \to X$, i.e., $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$, still denoted as $X \simeq Y$.

[The notation $\simeq$ can be confusing. See whether it's between maps or spaces from context.]

Claim: it is an equivalence relation.

Back to the deformation retraction picture. The package there gives $f_0 = Id_X \simeq f_1$, in fact $f_0 = Id_X \simeq_A f_1$.

Claim: if A is a deformation retract of X, then $X \simeq A$. In fact, $X \simeq_A A$.

Proof: the maps are $f_1 : X \to A \subset X$ and the inverse $g : A \to X$ being the inclusion of A back into X. Then we have

$f_1 \circ g = Id_A$ and $g \circ f_1 = f_1 \simeq_A Id_X = f_0$. That's it.

X is **contractible** if X is homotopy equivalent to a point, which can always be taken to be in X. The maps have to be

$$f : X \to \{p\}, \quad g : \{p\} \to X$$

The contractibility requires $g \circ f : X \to X \simeq Id_X$ with $g \circ f$ being a constant map. The other one is trivial as $f \circ g : \{p\} \to \{p\} = Id_{\{p\}}$.

[A priori, we might not have $\{p\}$ as a deformation retraction of X since we do not require $g(p) = p$ and $f \circ g \simeq_{\{p\}} Id_X$. The former one certainly superficial as we could always identify p with the image $g(p)$. The latter one is less clear by definition. However, by the later discussion about the Homotopy Extension Property, we know it is true when $(X, \{p\})$ satisfies this property, which is the case for a CW pair. So this is almost always the case.]

Remark: the construction of homotopy equivalence between spaces is sometimes tricky and tedious to write down in every detail. However, the idea behind is usually quite intuitive. Just as the deformation retraction between X and A, one needs to find a way to squeeze or stretch from one to the other in a continuous way. Also, it is frequently more convenient to prove two spacess are homotopic through a third object, using equivalence relation, for example, the homotopy equivalence between different type of glass frames on Page 2 in Hatcher.

Cell Complexes

Idea: this is a systematic way of building up complicated spaces from simple ones.

CW complexes consist of a large family of topological spaces which is enough in most cases. They are convenient enough to work with and general enough to attract extensive interest.

[They represent all topological spaces under the so-called "weak homotopy equivalence" which makes use of general homotopy groups.]

1. A simple example.

1-torus coming from a square by identifying opposite sides without twisting them.

2. General procedure.

We have the following construction:

Step 1: start with a set of discrete points X^0, the 0-skeleton, with points seen as 0-cells.

Step 2 (induction step): for $n \geqslant 1$, form n-skeleton X^n by "gluing" the boundary of each n-cell to the $(n-1)$-skeleton, X^{n-1}.

The resulting space is called a cell complex or CW (J. H. C. Whitehead) complex.

Clarification:

- For $n \geqslant 1$, n-cell is (topologically) open n-disk with boundary being S^{n-1}, denoted e^n. [Consider this in $\mathbb{R}^n$.]
- "gluing" is an intuitive way of describing the process and one can use the notion of quotient space to make it rigorous.

[Step 2 can stop after finite times with the resulting space (of finite dimension) having the usual topology (from quotient) or continue forever. For the latter case, we would give it the weak topology: a subset A is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for all n's. We focus on such spaces of finite dimension.]

Each n-cell has a **characteristic map** which is the boundary gluing map plus the inclusion map for the interior.

Denote D^n as the closed n-disk. Then the characteristic map is $D^n \to X^n \subset X$ with the gluing map $\partial D^n = S^{n-1} \to X^{n-1}$, and $e^n \to X^n$ being homeomorphic to the image.

3. Examples.

a) Graph.

b) Triangulation of Riemann surface.
[Euler characteristic and Euler formula, $V - E + F = 2 - 2g$.]

c) S^2 [CW complex structure is not unique in general.]
i) e^0 and e^2;
ii) e^0, e^1 and two e^2: equator and two hemispheres;
iii) two e^0, two e^1 and two e^2.

4. Subcomplex.

A subcomplex (of a CW complex) is the union of cells (using the characteristic map) and a closed subspace.

It is NOT just union of some cells. One can use the above CW structure of S^2, iii), to get counter-examples.

It (i.e., the closedness) guarantees A is a CW complex itself with those cells from X since the images of characteristic maps for all those cells are also contained in A.

The CW complex X and a subcomplex A form a CW pair (X, A).

Operations on Spaces (CW complexes)

1. Quotient space: $\sim$ is an equivalence relation for elements (i.e., points) in X, then we have a quotient space $X/\sim$ defined by the following properties:

i) as a set, it's the set of equivalence classes;

ii) open sets in $X/\sim$ are those with open "pre-images" in X [as in Hillman notes, it is exactly the topology making sure the natural map $X \to X/\sim$ is continuous].

Examples:

(a) $A \subset X$: by "collapsing" (or crushing) A to a point, we get X/A.

In particular, for a CW pair (X, A), X/A has an induced CW complex structure. [The closedness of A is important.]

[Compare: $X = S^2$ and A is the open (or closed) hemisphere.]

(b) $A \subset Y$, $f : A \to X$, then we have $X \sqcup_f Y$ (used in constructing CW complex) by "attaching" (or "gluing") Y to X via f.

(c) We finish with a concrete one: $S^1 \times \mathbb{R}_{\geqslant 0}/S^1 \times \{0\} = \mathbb{R}^2$, where as usual, equality stands for homeomorphic.

A handy tool in defining a (continuous) map from a quotient space: if $f : X \to Y$ is continuous and it makes sense as a map $F : X/\sim \to Y$ (i.e. points of X in the same equivalence class got mapped to the same point in Y) , then F is continuous.

2. Product space and product CW complex

We mostly work with CW complexes with finitely many cells.

Example: $T^2 = S^1 \times S^1$.

Extra care regarding product topology as space or CW complex for complexes with infinite cells.

The following are a bunch of constructions using quotient.

3. Cone and suspension.

For X, the cone CX is $X \times I$ with $X \times \{0\}$ collapsed to a point. (It is always "contractible"!)

For X, the suspension SX is $X \times I$ with $X \times \{0\}$ and $X \times \{1\}$ collapsed to two points. (Union of two cones.)

Examples: $CS^1 = D^2$ and $SS^1 = S^2$.

4. Wedge product: $X \vee Y$ is X and Y attached together at one point.

Example: $S^1 \vee S^1$ is the figure eight.

Question: does the choice of points matter?

5. Join and smash product. [Hatcher, Pages 9-10.]

Join: $X * Y = X \times Y \times I / \sim$, where
$(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$.

Intuitively, that is the set of all convex combinations,
$(1 - t)x + ty$, for $x \in X, y \in Y, t \in [0, 1]$.

Smash product: $X \wedge Y = X \times Y / X \vee Y$, where
$X = X \times \{y_0\} \subset X \times Y$ and $Y = \{x_0\} \times Y \subset X \times Y$ for $X \vee Y$.

Examples:
i) $S^m * S^n = S^{m+n+1}$ (using coordinate);
ii) $S^m \wedge S^n = S^{m+n}$ (using product CW complex construction).

Two Criteria for Homotopy Equivalence

1. Collapsing subspaces.

Result: for a CW pair (X, A) with A contractible, the quotient space $X \to X/A$ is a homotopy equivalence.

Explanation: we provide the proof later, making use of the homotopy extension property for a CW pair (X, A).

Example: collapse one closed hemisphere of S^2 to also get S^2.

In general, homeomorphism is too much to ask for. More examples in Hatcher.

2. Attaching spaces.

Result: for a CW pair (Y, A), and two attaching maps $f, g : A \to X$, if $f \simeq g$, then $X \sqcup_f Y \simeq X \sqcup_g Y$.

Explanation: we provide the proof later, making use of the homotopy extension property for a CW pair (X, A).

Examples:

i)) For $X \wedge Y$, the choice of gluing points doesn't matter within path-connected components in X.

ii) S^2 and "S^2 with folds".
X is the lower hemisphere, Y is the upper hemisphere, both being closed and A is the equator for Y. A can have small pieces to go back and forth when gluing to the equator of X, causing "folds". It requires some work to see S^2 and "S^2 with folds" are not homeomorphic, by checking locally at folded area, however, they are homotopic by this result.

Homotopy Extension Property

1. **Definition:** $A \subset X$, if for any $f_0 : X \to Y$ with a homotopy $f_t : A \to Y$ of the restriction $f_0 : A \to Y$, the homotopy f_t can be extended to $X \to Y$ for $f_0 : X \to Y$, then say (X, A) has **homotopy extension property**.

[This is for any choice of Y, f_0 and f_t.]

Equivalently, it means any (continuous) map $X \times \{0\} \cup A \times I \to Y$ can be extended to $X \times I \to Y$.

Further equivalently, one just needs a map $X \times I \to X \times \{0\} \cup A \times I$ extending the identity map $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$.

In the last equivalent definition we call $X \times \{0\} \cup A \times I$ a **retract** of $X \times I$, which is weaker than the notion of deformation retract.

In general, $A \subset X$ is a **retract** means there is a retraction map $r : X \to A$ such that $r \circ i = Id_A$ where $i : A \to X$ is inclusion.

[This is certainly weaker than deformation retract, with no requirement of a homotopy relative to A between the retraction map and Id_X. Examples to illustrate this point are easy to construct as there might be other path-connected components in X not intersecting A.]

2. Some Results.

a) A CW pair (X, A) has homotopy extension property. In fact, $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$ in this case.

[So Homotopy Extension Property is quite common.]

Explanation: there is the explicit construction of a deformation retraction from $D^n \times I$ to $(D^n \times \{0\}) \cup (S^{n-1} \times I)$ for $n \geqslant 1$. When $n = 0$, it's trivially true.

This construction can be used to get rid of (by deformation retraction) all cells in $X \setminus A$.

b) If (X, A) has homotopy extension property and A is contractible, then $q : X \to X/A$ is a homotopy equivalence.

The combination of Id_X and the contracting homotopy of $A \times I \to A$ can be extended to $F : X \times I \to X$, inducing the homotopy inverse $f : X/A \to X$ from $F(\cdot, 1) : X \times \{1\} \to X$. This also shows $Id_X = F(\cdot, 0) \simeq F(\cdot, 1) = f \circ q$.

This homotopy $F : X \times I \to X$ induces $\widetilde{F} : X/A \times I \to X/A$, justifying $Id_{X/A} = \widetilde{F}(\cdot, 0) \simeq \widetilde{F}(\cdot, 1) = q \circ f$.

c) (X, A) has homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X. [Proof is not required.]

[For the other direction, we know that deformation retraction implies homotopy equivalence. It's good to see that these two notions are almost equivalent.]

d) For a CW pair (Y, A), and two attaching maps $f, g : A \to X$, if $f \simeq g$, then $X \sqcup_f Y \simeq X \sqcup_g Y$.

Explanation: this makes use of the above fact a), i.e., $Y \times I$ can be deformation retracted to $(Y \times \{0\}) \cup (A \times I)$, which is stronger than the homotopy extension property of the pair (Y, A).

Let $F : A \times I \to X$ gives the map homotopy between f and g, and $G : (Y \times I) \times I \to Y \times I$ gives the deformation retraction from $Y \times I$ to $(Y \times \{0\}) \cup (A \times I)$.

They induce the following map:

$$\big(X \sqcup_F (Y \times I)\big) \times I \to X \sqcup_F (Y \times I)$$

by defining it to be identity over X. Here it is important that the deformation retraction map fixes the gluing part $A \times I$.

This provides a deformation retraction of $X \sqcup_F (Y \times I)$ to $X \sqcup_F \big((Y \times \{0\}) \cup (A \times I)\big)$.

We observe that $X \sqcup_F \big((Y \times \{0\}) \cup (A \times I)\big) = X \sqcup_f Y$, and so $X \sqcup_F (Y \times I) \simeq X \sqcup_f Y$.

Similarly, $X \sqcup_F (Y \times I) \simeq X \sqcup_g Y$. Hence $X \sqcup_f Y \simeq X \sqcup_g Y$.

Suggested Problems

1. Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}. [Hatcher, Chapter 0, Exercise 2]

2. Show that homotopy equivalences between maps and spaces are both equivalence relations.

3. Suppose $f, g : S^{n-1} \to X$ are homotopic maps. Show by explicit construction that $X \sqcup_f D^n$ and $X \sqcup_g D^n$ are homotopic to each other. [Hillman notes, Homology, Exercise 14]

[This is an example on the result stated for "attaching spaces" as one of the two criteria for homotopy equivalence.]

4. Let $f : S^{n-1} \to X$ be a map and $g : X \to Y$ be a homotopy equivalence. Show that $X \sqcup_f D^n \simeq Y \sqcup_{g \circ f} D^n$. [Hillman notes, Homology, Exercise 15]

5. $X = S^1 \vee S^1$ and Y is the space obtained by connecting two S^1's with an interval.

1) Show X and Y are homotopic to each other.

2) Show X and Y are not homeomorphic to each other.

6. $S^1 \wedge S^1 = S^2$ and $S^1 * S^1 = S^3$.

Chapter 1: Fundamental Group

Motivation

Classification problem and topology (homotopy) invariants.

Example: how to tell apart a disk and a punctured disk?

A disk "is" a point. [contractible, i.e. homotopy type of a point, in fact a deformation retract here]

A puncture disk "is" a circle. [the notion of deformation retract]

Let's be brave to say a point and a circle are different.

Mission accomplished. Certainly, we want to make it rigorous.

Similarly, are open and closed balls the same (homeomorphic)?

This actually illustrates the importance of loops in a space X. It is the space of them that we are going to focus on now and the information squeezed out is the "fundamental group" of X.

Section 1.1 Basic Construction

Paths and Homotopy

1. Notions of path.

Path in X: a map $f : I \to X$, not necessary close path (i.e. loop), which means it is not necessary to have $f(0) = f(1)$.

Homotopy of paths with the same endpoints (initial and terminal, i.e. counting order): a family of maps $f_t : I \to X$ between paths f_0 and f_1 with $f_t(0) = f_0(0) = f_1(0)$ and $f_t(1) = f_0(1) = f_1(1)$, denoted as $f_0 \simeq f_1$.

More specifically, $F : I \times I \to X$ with $F(0,t) = f_0(0) = f_1(0)$, $F(s,1) = f_0(1) = f_1(1)$, $F(s,0) = f_0(s)$ and $F(s,1) = f_1(s)$, where s is path parameter and t is homotopy parameter.

Note:

i) path is the map, not just the image;

ii) path homotopy, with the restriction on endpoints, is stronger than homotopy of maps.

iii) path homotopy is equivalence relation among paths with the same endpoints, and so we have equivalence class $[f]$ for the set of paths with the same endpoints as $f : I \to X$, called **homotopy class** of f .

Path product or composition: for two paths f and g, if $f(1) = g(0)$, one can easily have $g \cdot f : I \to X$ with the first half being f and second half being g.

More explicitly, $g \cdot f(t) = f(2t)$ for $t \in [0, \frac{1}{2}]$ and $g(2t-1)$ for $t \in [\frac{1}{2}, 1]$.

Note: the order we use is reversed comparing with the choice in [Hatcher]. Of course, this is a rather superficial difference.

Properties:

a) if $f_0 \simeq f_1$, $g_0 \simeq g_1$ and $f_0(1) = g_0(0)[= f_1(1) = g_1(0)]$, then $g_0 \cdot f_0 \simeq g_1 \cdot f_1$, and so $[g] \cdot [f] = [g \cdot f]$ is well defined [if the composition can be done];

b) $[g \cdot f]$ doesn't depend on the choice of ratio (half-half for our choice).

[For path homotpy, the illustrative picture used before with square standing for $I \times X$ becomes more practical.]

2. Fundamental group.

This product or composition of homotopy class is not so convenient because of the requirement on endpoints, but it is already satisfying if we restrict to loops in X with a fixed basepoint (another name of endpoints in this case).

The set of homotopy classes of loops in X based on x_0 is denoted by $\pi_1(X, x_0)$. It can be called the **fundamental "group" of X at the basepoint** x_0 if we can prove the following claim.

Claim: $\pi_1(X, x_0)$ *is a group under the product introduced before.*

Proof: there are a few things to check for groups.

i) identity, **constant loop** c;

ii) inverse, **reversed loop** $\bar{f}$ **for** f;

[For any path f not necessary loop, $\bar{f} \cdot f \simeq c$. The idea is to turn around earlier. $f(s)$ has "homotopy" $f((1-t)s)$ and $\bar{f}(s)$ has "homotopy" $\bar{f}(1-(1-t)(1-s))$. Then we have the path composition as the desired homotopy. The turning point is $f(1-t) = \bar{f}(t)$.]

iii) associativity relates to the independence of the ratio in path product mentioned earlier.

Note: the lower index 1 in $\pi_1(X, x_0)$ comes from the view of homotopy class of maps $S^1 \to X$ with a fixed basepoint, as part of a more general homotopy group theory which is beyond the scope of this course.

3. Basic properties.

a) Choice of basepoint.

The choice of basepoints is not essential by staying in the same path component.

Construct (group) isomorphism $F : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by conjugating with the path between x_1 and x_0.

[One could construct the inverse homomorphism in the same way to prove it's isomorphic.]

[This isomorphism depends on the choice of homotopy class for the path between x_0 and x_1. To view on $\pi_1(X, x_0)$, it's not necessary the identity map but might be up to conjugation by a group element.]

So the notation is simplified to $\pi_1(X)$ if X is path-connected. We implicitly means that by writing $\pi_1(X)$ since it is ambiguous otherwise.

b) Simply-connected.

X simply-connected: $\pi_1(X) = 0$ (i.e. trivial, maybe $\{1\}$ is a better of notation since we call the group operation "product"). Of course, X is implicitly assumed to be path-connected.

Claim: X is simply-connected iff there is "a unique" (indicating path-connectedness of X) homotopy class between any two ordered points.

Proof:

$\Rightarrow$

for any two paths f and g between any two points,

$$f \simeq f \cdot \bar{g} \cdot g \simeq g,$$

where the second $\simeq$ makes use of $f \cdot \bar{g} \simeq c$ since $\pi_1(X) = 0$.

$\Leftarrow$

The simplest proof: just pick the ordered set to be of the same point p as the basepoint of the fundametal group.

If we want to see more intuition, for any loop based at x_0, it can be seen as the product of two paths $g \cdot f$, where $f \simeq \bar{g}$ with corresponding endpoints by assumption. It is easy to get a homotopy for $g \cdot f$ to c from the homotopy between f and $\bar{g}$.

[One can either deal with the constant map sides of that $I \times I$ by another homotopy or use quotient to get rid of them.]

That's it!

One can play around more about this, for example, (Alex Casella)

Claim: for a path-connected X, X is simply-connected iff there exists $p, q \in X$ which can be the same with a unique path homotopy class from p to q.

Let's only see the "$\Leftarrow$" part.

Consider any $[l] \in \pi_1(X, p)$. f is a path from p to q.

Then $f \cdot l \simeq f$. So $l \simeq \bar{f} \cdot f \cdot l \simeq \bar{f} \cdot f \simeq c_p$.

c) $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, where implicitly X and Y are assumed to be path-connected. [Hatcher Prop. 1.12]

Proof: clear from definition.

[Recall the meaning of group product.]

[Again, the isomorphism can be justified by having explicit inverse homomorphism.]

Example:

$\pi_1(T^2) = \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ using the result from the next topic.

Remark: easy to see directly that fundamental group is a topological invariant, i.e. $X = Y$ (homeomorphic) implies $\pi_1(X) = \pi_1(Y)$. In fact, $X \simeq Y$ would be enough. We'll have more discussion on this later. By reducing a space X to its fundamental group $\pi_1(X)$, we make the information package a lot more compact.

Fundamental Group of Circle

1. Consider S^1 as the unit sphere in $\mathbb{R}^2$.

Why S^1? Well, we only need to consider path-connected spaces. Clearly $\pi_1(\{p\}) = 0$, and the next simplest space is S^1.

Theorem: the map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ defined by

$$\Phi(n) = [\omega_n = (\cos(2\pi ns), \sin(2\pi ns))]$$

is an isomorphism between groups.

Remark: [Hatcher] provides the great detail of a rigorous proof by lifting a loop $I \to S^1$ and relevant homotopy to $I \to \mathbb{R}$.

[There is the idea of covering space hidden behind which we would focus on later. We'll say something about this argument then.]

Here, instead we provide an alternative approach which is more intuitive and less rigorous. Essentially, they are the same.

Key Idea: we can reduce all loops to be based at $(1, 0) \in \mathbb{R}^2$ (for $s = 0$ and 1) and reach the basepoint only finitely many times and sometimes stays there for a while. In fact, we could make the representatives *"piecewise of constant speed"*.

[The crazy small parts would stay in a small piece ("not going around the circle") and so can be made more regular by continuous deformation, i.e., homotopy. Clearly, one can also use homotopy to eliminate the "stays".]

[Uniform continuity can be used to make it rigorous. One could find enough points on the path so that the genuine path between consecutive ones, being contained in a small piece on S^1, is homotopic to the constant speed path between them.]

More Description: each of the piecewise constant-speed path is clearly homotopic to one of the ω_n.

Here is a constructive way to get this n. Let's count the directions, $+$ and $-$ for counter-clockwise and clockwise, and get an ordered set $\{a_i\}$ with $a_i \in \{\pm 1, \pm\frac{1}{2}\}$ and i being the index counting the loop passing the basepoint (in directions) of reaching and leaving (with directions).

[One could also break each "passing the basepoint" by arriving and leaving, and so replace 1 by two halves.]

Consecutive $+$ and $-$ would cancel each other by homotopy. (Why? Easy intuition by deforming path.)

So one can **sum** them up and get an integer (with even times of "leaving" and "arriving" in all). This is enough to justify group homomorphism and surjectivity.

We also observe that loop homotopy could only change the set $\{a_i\}$ by cancelling or adding consecutive $+$ and $-$ while keeping the same sum, i.e., ω_n. This would be enough to see injectivity.

More precisely, one needs to see that the choice of the piecewise constant-speed loop representative for any loop won't affect the sum. This can be done by taking a common refinement of any two piecewise constant-speed path representatives, and then see that they all give the same sum.

Remark: indeed, the construction above gives the inverse of Φ.

For the injectivity part, if the intuition is too "dodge", one can also prove the trivial kernel for Φ by applying a little homology theory discussed later in this semester.

2. Applications.

a) **Fundamental Theorem of Algebra:** nonconstant $\mathbb{C}$ polynomial has one solution in $\mathbb{C}$.

Proof: suppose $p(z) = z^n +$ lower degree terms is the polynomial of no solution.

For any real $r \geqslant 0$,

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

gives a loop in S^1 based at $1 \in \mathbb{C}$. f_0 is the constant loop, and so all f_r are homotopic to the constant loop.

Consider $p_t(z) = z^n + t\cdot$lower degree terms. For large r, $p_t(re^{2\pi is})$ is nonzero. So by the same construction of f_r, using p_t, the original f_r (for $t = 1$) is homotopic to the n-round loop (for $t = 0$), which is not trivial. *Contradiction!*

b) **Brouwer Fixed Point Theorem:** every continuous map $f : D^2 \to D^2$ has a fixed point x, i.e., $f(x) = x$.

[Notations: e^n is the open n-disk and D^n is the closed n-disk.]

Proof: prove by contradiction. Otherwise, use the ray from $h(x)$ to x to get a map $H : D^2 \to S^1$ fixing the boundary.

However, this would provide a homotopy from one round loop to constant loop in S^1.

[This is a classic construction which can be generalized. Not constructive, not satisfying?]

c) **Borsuk-Ulam Theorem in Dimension 2:** every continuous map $f : S^2 \to \mathbb{R}^2$ has two antipodal points x and $-x$ on S^2 with the same image.

Proof: prove by contradiction. Otherwise, $F(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ mapping S^2 to S^1.

There is a key feature that $F(-x) = -F(x)$.

If one considers any big circle in S^2 as a loop, this property indicates that the image loop in S^1 under F would be nontrivial (odd number of rounds) since intuitively the path has to move in a way symmetric to the first half of time with respect to the center, and this creates one extra lap, making the total number odd.

[Lifting argument as in Hatcher can be used to make it rigorous.]

However, the map F would provide a homotopy to the constant loop. *Contradiction!*

One Very Intuitive Implication: for $S^2 = A_1 \cup A_2 \cup A_3$ with all A_i's closed sets, (at least) one of A_i contains a pair of antipodal points.

Proof: $d_i : S^2 \to \mathbb{R}$ defined as $d_i(x) = \inf_{y \in A_i} |x-y|$, and we have a map

$$f : S^2 \to \mathbb{R}^2, f(x) = (d_1(x), d_2(x)).$$

By Borsuk-Ulam Theorem, one has $x \in S^2$ such that $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$.

Either one being 0 means x and $-x$ are in that set. Otherwise, they should both be in A_3.

Induced Homomorphism

1. A map $f : X \to Y$ clearly induces $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ by tranferring loop and homotopy.

If X is path-connected, then the choice of x (and so $f(x)$) would not matter if taking the conjugations into account and end up with a commuting diagram ("a square"). The path in Y comes from the chosen path in X.

If furthermore, Y is also path-connected, then one can use other point in Y other than $f(x)$ using conjugation. So we can write $f_* : \pi_1(X) \to \pi_1(Y)$ without confusion.

Basic Properties:
a) $Id : X \to X$ induces $Id_* : \pi_1(X, x) \to \pi_1(X, x)$.
b) $f : X \to Y$ and $g : Y \to Z$, then
$g_* \circ f_* = (g \circ f)_* : \pi_1(X, x) \to \pi_1(Z, g \circ f(x))$. [functor property]

Note: they are enough to show homeomorphic spaces have the same (isomorphic) fundamental group, i.e. $\pi_1(X)$ is a topological invariant.

2. More properties.

1) $\pi_1(S^n) = 0$ for $n \geqslant 2$
Intuitively, this is very obvious (as for S^2).
For a rigorous proof, just need to see each loop can be reduced to something not filling in the whole sphere as $S^2 \setminus \{p\} = \mathbb{R}^2$ and then use the inclusion map $i : \mathbb{R}^2 \to S^2$ if like. This is easy by homotopy.
[Space-filling curve is not a problem here. Consider $l : I \to S^n$ and $\overline{B}_\delta \subset B_{2\delta}$. $l^{-1}(B_{2\delta})$ might consist of infinite many (open) intervals, but $l^{-1}(\overline{B}_\delta)$ only sits in finitely many of them. One can easily deform the path out of $\overline{B}_\delta$.]

2) If $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic with homotopy map f_t, then the induced homomorphisms $f_{0*} : \pi_1(X, x) \to \pi_1(Y, f_0(x))$ and $f_{1*} : \pi_1(X, x) \to \pi_1(Y, f_1(x))$ are the same if take the conjugation by the path $f_t(x)$ from $f_0(x)$ to $f_1(x)$ into account.
["Same" here indicates a commutative diagram.]
Of course, if the homotopy f_t fixes the image of x (and then obviously $f_0(x) = f_1(x)$), we have $f_{0*} = f_{1*}$.

Proof: $X \times I \to Y$ gives $I \times I \to Y$ for any loop under consideration. For the square $I \times I$, one can easily deform one side to the composition of the other three, justifying that the map between $\pi_1(Y, f_0(x))$ and $\pi_1(Y, f_1(x))$ is the conjugation by that path.

3) $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is isomorphic for any $x \in X$.

Proof: there is $g : Y \to X$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$.
We have $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ and $g_* : \pi_1(Y, f(x)) \to \pi_1(X, g(f(x)))$, such that
$g_* \circ f_* : \pi_1(X, x) \to \pi_1(X, g(f(x)))$ the same as $Id_{X*} = Id : \pi_1(X, x) \to \pi_1(X, x)$. [the commutative diagram in 2, 2) in this topic]
So f_* is injective.

Also for $g_* : \pi_1(Y, f(x)) \to \pi_1(X, g(f(x)))$ and $f_* : \pi_1(X, g(f(x))) \to \pi_1(Y, f(g(f(x))))$ (the same as the f_* before [as in 1 of this topic], (II)), we have $f_* \circ g_* : \pi_1(Y, f(x)) \to \pi_1(Y, f(g(f(x))))$ the same as $Id_{Y*} = Id : \pi_1(Y, f(x)) \to \pi_1(Y, f(x))$, (I).
So f_* is surjective.

[The "same" above always means conjugating by the corresponding path from map homotopy, although the diagram might commute for different reasons. In fact, here map between $\pi_1(X, f(g(f(x))))$ and $\pi_1(X, f(x))$ appears twice in different directions, making use of (I) the path for $f(x)$ under the homotopy $f \circ g \simeq Id_Y$, and (II) the image path under f of the path for x under the homotopy $g \circ f \simeq Id_X$.]
So f_* is isomorphic.

4) If A is a retract of X, then the inclusion $i : A \to X$ induces injective $i_* : \pi_1(A, x) \to \pi_1(X, x)$.
If A is a deformation retract of X, then the inclusion $i : A \to X$ induces isomorphic $i_* : \pi_1(A, x) \to \pi_1(X, x)$.

Proof: [We do not have the basepoint issue here.]
There is $r : X \to A$ such that $r \circ i = Id_A$.
Choose a basepoint $x \in A$ and we have
$r_* : \pi_1(X, x) \to \pi_1(A, x)$, $i_* : \pi_1(A, x) \to \pi_1(X, x)$.
Then $r_* \circ i_* = Id_{A*} = Id$. Hence i_* is injective.

If A is a deformation retract, then there is a homotopy between $Id_X : X \to X$ and $i \circ r : X \to A \to X$, fixing A.
So $i_* \circ r_* = Id_{X*} = Id$. Hence we conclude that i_* is surjective and so an isomorphism.

5) $\mathbb{R}^2 \neq \mathbb{R}^n$ for $n \neq 2$.

Proof: otherwise, $\mathbb{R}^2 \setminus \{p\} = \mathbb{R}^n \setminus \{q\}$.

$n = 1$: $\mathbb{R}^1 \setminus \{p\}$ is not path-connected, while the others are.

For $n > 1$, $S^1 \simeq S^{n-1}$ contradicts with fundamental group imformation.

[Being contractible, all $\mathbb{R}^n$'s are homotopic.]

Suggested Problems

1. (Cancellation Property for Path Composition) For paths f_i and g_i, $i = 0, 1$, with proper endpoints (for the compositions to make sense). Assume $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$. Prove (by definition) $f_0 \simeq f_1$.

[Hatcher, Section 1.1, Exercise 1]

2. For any space X, show that the following are equivalent:

i) every map $S^1 \to X$ is homotopic to a constant map (i.e. (i.e. the image containing just one point));

ii) every map $S^1 \to X$ extends to a map $D^2 \to X$;

iii) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

[Hatcher, Section 1.1, Exercise 5]

3. Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ is an induced homomorphism of a map $S^1 \to S^1$.

[Hatcher, Section 1.1, Exercise 12]

4. Use explicit construction to justify the commutativity of the following two elements in $\pi_1(X \times Y, x \times y)$, $[F]$ and $[G]$ with $F(s) = (f(s), y)$ and $G(s) = (x, g(s))$, where f and g are loops in X based at x and in Y based at y respectively.

[Hatcher, Section 1.1, Exercise 10]

Section 1.2 Van Kampen Theorem

Idea and Preparation

1. How to compute fundamental group for a space X?

Using definition itself is not always that pleasant, for example, S^1 is a challenge already. [One comparison: how often do you use the definition of definite integral to compute it?]

We already have something about $X \times Y$, but it won't get us too far by itself, for example, not helpful for the figure 8, i.e. $S^1 \vee S^1$.

The idea is to break a space into smaller (and simpler) pieces.

Intuitively, $\pi_1(S^1 \vee S^1)$ should have two generators, with little to no relation. [Picture this!]

2. Let's review some algebra notions first.

For a collection of groups $\{G_\alpha\}$ with α in some index set, finite or infinite, we have:

Product group: $\Pi_\alpha G_\alpha$,
vectors with possibly infinitely many nontrivial components.

Direct sum group: $\oplus_\alpha G_\alpha$,
vectors with finitely many nontrivial components.

Remark: the unit for either one is the "vector" with all components being trivial (i.e. unit). Product and sum are conventional choice of terminology. They are the same if the index set is finite. The group operation is done for each component separately. More abstractly [functor property],

i) given homomorphisms $H \to G_\alpha$, always have $H \to \Pi_\alpha G_\alpha$, for any group H.

ii) given homomorphisms $G_\alpha \to H$, always have $\oplus_\alpha G_\alpha \to H$, for any commutative group H.

iii) $\oplus_\alpha G_\alpha$ is a (normal) subgroup of $\Pi_\alpha G_\alpha$.

iv) The overall "vector space" structure provides some commutativity.

Free product group: $*_\alpha G_\alpha$, group of (reduced, i.e. identities of these groups ignored) words of finite length. Unit is the empty word.

Obviously, there is a homomorphism $*_\alpha G_\alpha \to \oplus_\alpha G_\alpha$ with empty word mapped to the unit. Also, given homomorphisms $G_\alpha \to H$, always have $*_\alpha G_\alpha \to H$, for any group H.

Note: pay attention to the confusing names of group operation. The same one might be called summation or product as conventional choices.

Example:

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ has 4 elements with unit being $(0, 0)$.

$\mathbb{Z}_2 * \mathbb{Z}_2$ has infinite elements: two pieces of information to decide a nontrivial element, the starting letter with 2 choices and the length.

Van Kampen Theorem

Theorem: Suppose $X = \cup_\alpha A_\alpha$ with each A_α is a path-connected open set and containing x_0.

i) If each intersection $A_\alpha \cap A_\beta$ is path-connected, then the natural homomorphsim $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.

ii) If each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the above homomorphism Φ is surjective with the kernel being the normal subgroup, N, generated by $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ where $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$, and so $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N$.

Explanation of the Statement:

a) X is path-connected for sure;

b) $\cap_\alpha A_\alpha$ contains x_0 and so is non-empty;

c) no need to assume the indices α, β and γ being different;

d) recall the definition of normal subgroup. Kernel of a homomorphism is always normal, and also the quotient would be a group;

e) quotient by the normal group N generated by $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ simply means the loop in the intersection of two A_α's can be viewed as in either one.

Idea of the proof:

Surjectivity of Φ is not hard by breaking a loop into loops in each A_α, where the connectivity of $A_\alpha \cap A_\beta$ is curcial.

For the statement about N in ii), obviously it should contain all such elements.

For the other direction, one needs to look at the homotopy between two loops more carefully to see the change of representatives in the free product has to come from things like $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ [as in Hatcher, Pages 45–46]. Connectivity of intersections of at most three sets is needed for this.

More precisely, each homotopy "rectangle" can be broken into small rectangles each mapped to some A_α, with each vertex on the boundary of at most three of them. Then each vertex can be connected to x_0 by a path in all three relevant A_α's. Each edge (with only two vertices as endpoints) of the small retangles can now be viewed as a loop in all those A_α. One could then write the homotopy in "N" by moving over each small rectangle one-by-one.

Examples as application:

Note: A_α's needs to be open. However, in practice, this could be "avoid" frequently.

(1) $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

(2) How about X being three circles attached together, not at the same point?

Plan A: homotopic to wedge sum of three circles, with both being deformation retracts of a space with an additional 2 dimensional piece. [Apply theorem once for three-set union.]

Plan B: one can also apply Van Kampen Theorem directly. [Apply theorem twice for two-set union.]

(3) $\pi_1(S^2) = 0$: "two hemispheres".

(4) $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$: "1-skeleton and the 2-cell".

Applications to Cell Complexes

1. Attaching 2-cells.

Result: the space Y is constructed by attaching a 2-cell, e^2, to a path-connected space X via $f : S^1 \to X$. Then $\pi_1(Y) = \pi_1(X)/N$ with the normal subgroup N (essentially) generated by $[f]$.

Proof: clearly Y is also path-connected.
Suppose the basepoint is on the image of f (for the "essentially" in the statement).
To apply Van Kampen Theorem, we have two (open) sets, "X" and "D^2".
[There is some technical issue caused by the choice of basepoint, which can be dealt with by a more careful construction as in Hatcher, Page 50.]
$\pi_1(``D^2") = 0$ and the normal subgroup from the theorem is clearly generated by $[f]$.

Remark: in principle, we now have the complete machinery to compute $\pi_1(X)$ for 2-dimensional cell complexes, which includes the result on wedges of S^1's and the above one on attaching 2-cells.

2. Group as Fundamental Group.

Result: any group is the fundamental group of a 2-dimensional cell complex.

Proof: group is decided by generators and relations.
Generators are realized by S^1's in the wedge and relations are realized by attaching 2-cells accordingly.

Suggested Problems:

1. Let $X \subset \mathbb{R}^m$ be the union of convex open sets $\{X_i\}_{i \in E}$ for some index set E. Suppose $X_i \cap X_j \cap X_k \neq \varnothing$ for any i, j and $k \in E$. Prove that X is simply-connected.

[Essentially, Hatcher, Section 1.2, Exercise 2]

2. Show that the complement of finitely many points in $\mathbb{R}^n$ is simply-connected if $n \geqslant 3$.

[Hatcher, Section 1.2, Exercise 3]

3. Let $X \subset \mathbb{R}^3$ be the union of n lines through origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

[Hatcher, Section 1.2, Exercise 4]

4. Let X be the quotient space of S^2 by identifying north and south poles as one point. Give X an explicit cell complex structure and use it to compute $\pi_1(X)$.

[Hatcher, Section 1.2, Exercise 7]

Section 1.3 Covering Space

Idea and Definition

1. The presence of (large) fundamental group indicates complication in the topology of a space. However, sometimes, we do not want to include it in the study of this space (for example, local information is sometimes more focused on in the field of Differential Geometry), or a proper point of view could elimination the affect of this data.

For example, $\mathbb{R}$ would be enough for S^1 is we only care about the distance traveled in which direction. We have simple motion on $\mathbb{R}$ would cover S^1 as many times as possible. More precisely, there is the map $f(x) = e^{2\pi i x} : \mathbb{R} \to S^1$. Locally, they are indeed identical.

2. **Definition:** a covering space of X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying that for each point $x \in X$, there is an open neighbourhood U with $p^{-1}(U)$ being the union of disjoint open sets V_i's, with the restrictions $p : V_i \to U$ all being homeomorphic.

p is called the covering map.

The number of V_i can be called the number of **sheets** for the covering space at the point p, which is also the cardinality of $p^{-1}(x)$ for any $x \in V$.

It is very easy to see that for path-connected X (or each path-connected component of X in general), it is a constant (positive integer) not depending on the choice of p.

Warning: the same $\widetilde{X}$ might have different covering maps p's. $\widetilde{X}$ might not be path-connected for path-connected X.

Examples:

a) Considering S^1 as the unit sphere in $\mathbb{C}$, covering maps $S^1 \to S^1 : z \to z^n$ for any integer n.

b) Covering spaces of $S^1 \vee S^1$ as in Hatcher, Pages 57–58.

Universal Cover: for path-connected X, the package $(\widetilde{X}, p)$ with $\widetilde{X}$ simply-connected is called a universal cover.

Remark: it is "unique" and covers all the other covering spaces. The $\mathbb{R} \to S^1$ appearing before is an example. This is discussed later in detail.

Lifting Properties

Idea: "lifting" means getting a map $\widetilde{f} : Y \to \widetilde{X}$ for $f : Y \to X$ compatible with the covering map $p : \widetilde{X} \to X$, quite a natural choice of terminology.

There are three properties for us about "lifting".

1. Homotopy Lifting Property (or Covering Homotopy Property): for a covering space $p : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$ and the lift $\widetilde{f_0} : Y \to \widetilde{X}$ for (the initial map) f_0, there is a unique lift $\widetilde{f_t} : Y \to \widetilde{X}$ for f_t.

Idea of Proof:

Firstly, construct the lift of homotopy locally over Y, which is clear by the local homeomorphic property of $p : \widetilde{X} \to X$.

Secondly, the uniqueness is clear by looking at each point of Y since by the local homeomorphic property of p, there is only one choice.

Finally, the uniqueness makes sure the local lifts of homotopy can be glued together to get $\widetilde{f_t}$.

Applications:

To begin with,

i) taking Y to be the space of just one point, one can see any path in X can be lifted to a unique path in $\widetilde{X}$ under "any" choice of the initial point.

ii) taking Y to be I, one can see any path homotopy of path in X can be lifted to a path homotopy in $\widetilde{X}$ with respect to any lift of the initial path. Here the endpoints are fixed in the lift because of the uniqueness of Homotopy Lifting Property, i.e. applying i) to each endpoint.

They have implication on fundamental groups.

Result: given a covering space $p : \widetilde{X} \to X$, the map $p_* : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x)$ with $p(\widetilde{x}) = x$ is injective. The image subgroup consists of homotopy classes of loops in X based at x with loops as lifts in $\widetilde{X}$.

Idea of Proof:

Injectivity is by lifting the loop homotopy in X.

The characterization of image makes use of the uniqueness of lifting loop. [Is it a normal subgroup? Discussed later.]

This further relates to the number of sheets for the covering space.

Result: for path-connected $\widetilde{X}$, the number of sheets for the covering space $p : \widetilde{X} \to X$ is equal to the index of $p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ in $\pi_1(X, x)$, $[\pi_1(X, x) : p_*(\pi_1(\widetilde{X}, \widetilde{x}))]$. [In case of them being infinite, "equal" means one to one correspondent.]

Idea of Proof:
Let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}))$. $\Phi : \pi_1(X, x)/H \to p^{-1}(x)$ by looking at the terminal point of the lifted loop.
Well-defined: "fixing endpoints" for loop (or path) homotopy.
Surjective: $\widetilde{X}$ path-connected.
Injective: g_1 and g_2 have the same lifted terminal point, and so $\bar{g}_2 \cdot g_1$ is lifted to a loop in $\widetilde{X}$.

2. General Lifting Criterion (Existence): for a covering space $p : \widetilde{X} \to X$, $p(\widetilde{x}) = x$, and a map $f : Y \to X$, $f(y) = x$, with Y path-connected and locally path-connected. Then there is a lift $\widetilde{f}$ iff $f_*(\pi_1(Y, y)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}))$.

["Locally ..." means there is arbitrary small neighbourhood of each point satisfying "...". **Warning:** locally path-connected and path-connected do not imply each other in either direction. One can have a path-connected space with a point for which, any small neighbourhood has another point connecting to that point only by a long detour.]

Idea of Proof:
Starting with $\widetilde{f}$, it is easy by the induced homomorphism between fundamental groups.
Now to construct $\widetilde{f}$:
presumably, $\widetilde{f}$ is defined for each $z \in Y$ by lifting a path, coming from Y, in X from $f(y)$ to $f(z)$ with $f(y) = x$ lifted to $\widetilde{x}$ and picking up the terminal point as the image $\widetilde{f}(z)$;
$f_*(\pi_1(Y, y)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ makes sure the choice of such a path (orginally in Y) won't matter, and so $\widetilde{f}$ is well-defined;
locally path-connectedness means one can do things locally near z, with a fixed path from y to this small neighbourhood, justifying the continuity of the map constructed.

[This argument is by contruction, and so uniqueness is not clear from it. It is actually the case and discussed below.]

Example: "1. Homotopy Lifting Property" as a special case.

3. General Lifting Property (Uniqueness): given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$. If there are two lifts of f agreeing at one point of Y and Y is path-connected, then they are the same.

Idea of Proof:

Starting from the point these two lifts agree, the local homeomorphic property of the covering map make sure there is no room to wiggle.

[Alternatively, one can use "open, closed and non-empty" and "path-connected" to draw the conclusion.]

Classification of Covering Space

Motivation: it is always an interesting and important problem to characterize maths objects. Classification is the ultimate goal of that, providing a "list" (explicit or not) of all possibilities.

Classification Theorem: let X be path-connected, locally path-connected and semilocally simply-connected. Then there is a bijection Φ between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\widetilde{X}, \widetilde{x}) \to (X, x)$ and the set of subgroups of $\pi_1(X, x)$, mapping the covering space to the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ in $\pi_1(X, x)$.

By ignoring the basepoint, Φ is a bijection between the set of isomorphism classes of path-connected covering spaces $p : \widetilde{X} \to X$ and the set of conjugacy classes of subgroups of $\pi_1(X, x)$,

Explanation:

1) X is semilocally simply-connected: for each point in X, there is a neighbourhood U such that the homomorphism $i_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by inclusion is trivial. [Clearly, simply-connected is stronger than this.]

2) "Isomorphism class of covering spaces": two covering spaces are isomorphic if there is a homeomorphism between them covering the identity map for (the base space) X.

3) The existence and uniqueness of Lifting Property can be used to justify the injectivity of Φ.

4) For the surjectivity of Φ, one uses direct construction.

Define a universal cover $\widetilde{X} = \{[\gamma] \mid \gamma(0) = x, x \in X)\}$, $[\cdot]$ being path homotopy class, as a set and the covering map $p([\gamma]) = \gamma(1)$.

Define open sets (i.e. topology) as $U_{[\gamma]} = \{[\eta \cdot \gamma] \mid \eta(s) \in U, \eta(0) = \gamma(1)\}$, where U is an open set of X as in the definition of semilocally simply-connected.

Other covering space comes from taking quotient of $\widetilde{X}$ by equivalence relation generated by $[\gamma_1] \sim [\gamma_2]$ for $\gamma_1(1) = \gamma_2(1)$ and $[\bar{\gamma}_2 \cdot \gamma_1]$ in the subgroup.

5) Simply-connected covering space is unique (up to isomorphism) and it can be used to generate all other covering spaces, and so we call it the "universal" cover.

6) For the same $\widetilde{X}$ and p, a different choice of $\widetilde{x}$ would conjugate the image of p_*. It is not hard to incorporate this in 4).

Deck Transformation and Group Action

1. Basics.

Isomorphism (between covering spaces) from a covering space $\widetilde{X}$ to itself is more rigid (special) than homeomorphism, and is called *deck transformation* or *covering transformation*. They clearly form a group, called $G(\widetilde{X})$.

Basically, you are transferring between different sheets. Uniqueness of Lifting Property means deck transformation is decided by the image of one point (for $\widetilde{X}$ path-connected).

A covering space $p : \widetilde{X} \to X$ is called **normal** if for any two points in $\widetilde{X}$ over the same point in X, there is a deck transformation between them.

Well, this notion of "normal" clearly would have something to do with the subgroup $p_*(\pi_1(\widetilde{X}, x))$. We'll officially have it in a few minutes.

Example: $f : \mathbb{R} \to S^1$, $f(r) = e^{2\pi ir}$.

2. Results.

Result One: let $p : (\widetilde{X}, \widetilde{x}) \to (X, x)$ be a path-connected covering space of the path-connected and locally path-connected space X. Let H be the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x})) \in \pi_1(X, x)$. Then we have:

a) This covering space is normal iff H is a normal subgroup of $\pi_1(X, x)$;

b) $G(\widetilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x)$.

[Hence, if the covering space is normal, $G(\widetilde{X})$ is isomorphic to $\pi_1(X, x)/H$ with cardinality being the number of sheets. It is the case for the universal cover and $G(\widetilde{X})$ is isomorphic to $\pi_1(X)$.]

Idea of Proof:

To contruct deck tranformation, only need to justify the assumption for the existence part of General Lifting Criterion. Uniqueness part would guarantee that it is isomorphic.

Each element of $G(\widetilde{X})$, g, maps $\widetilde{x}$ to a point, $\widehat{x}$, (also) covering x. A path between $\widetilde{x}$ and $\widehat{x}$ gives an element $l(\widetilde{x}, \widehat{x}) \in \pi_1(X, x)$. The choice doesn't matter if take quotient with H.

Key: for the existence of deck transformation ("lift" of covering map), we need (and only need) to have $p_*(\pi_1(\widetilde{X}, \widetilde{x})) = p_*(\pi_1(\widetilde{X}, \widehat{x}))$.

Tracing back to the definitions of the map and loop homotopy class, these two subgroups are related by conjugation with $l(\widetilde{x}, \widehat{x})$. The equality means $l(\widetilde{x}, \widehat{x}) \in N(H)/H$.

So b) is proven.

Now we look at a). H being a normal subgroup means $N(H) = \pi_1(X, x)$, then $G(\widetilde{X})$ is one to one correspondent to $\pi_1(X, x)/H$ and so the set of points covering x.

[It might seem fishy to use one to one correspondence when both are infinite, but we actually have a bijection map.]

Now we introduce a more general setting of group action.

Group G acts on the space Y, i.e. there is a homomorphism from G to the homeomorphism group of Y, $Homeo(Y)$. One can quotient out the kernel and make sure it is an inclusion.

This action is *disjoint* if for each $y \in Y$, there is a neighbourhood U such that $U \cap g(U) \neq \varnothing$ iff g is the unit in G.

Result Two: If G acts on Y in a disjoint way, then

a) the quotient map $p : Y \to Y/G$ is a normal covering space;

b) G is the deck tranformation group of this covering space (if Y is path-connected for uniqueness of lifting);

c) G is isomorphic to $\pi_1(Y/G)/p_*(\pi_1(Y))$ if Y is path-connected and locally path-connected.

Idea of Proof:

Easy to see $p : Y \to Y/G$ is a covering space.

Easy to see by definition that G is the deck transformation group, and so (the normalizer is $\pi_1(Y/G)$ and so) this covering space is normal. So a) is proven.

c) is from b) of Result One.

Final Remark: simple speaking, the information about the fundamental group $\pi_1(X, x)$ is broken into two pieces, $\pi_1(\widetilde{X}, \widetilde{x})$ and the paths from $\widetilde{x}$ to the other points covering x.]

Suggested Problems

1. Let $p : \widetilde{X} \to X$ be a covering space with $p^{-1}(x) \neq \varnothing$ and being a finite set for any $x \in X$. Prove that

1) $\widetilde{X}$ is compact iff X is compact;

2) $\widetilde{X}$ is Hausdorff iff X is Hausdorff.

[Hatcher, Section 1.3, Exercise 3]

2. Suppose X is path-connected and locally path-connected, and $\pi_1(X)$ is a finite set. Prove that any (continuous) map $X \to S^1$ is nullhomotopic, i.e. homotopic to a constant map. [Hatcher, Section 1.3, Exercise 9]

3. Let $\widetilde{X}$ and $\widetilde{Y}$ be universal covers for path-connected and locally path-connected spaces X and Y respectively. Show that $X \simeq Y$ implies $\widetilde{X} \simeq \widetilde{Y}$, and use example to see the other direction of implication is not true. [Hatcher, Section 1.3, Exercise 8]

[Hint: do and apply Hatcher, Chapter 0, Exercise 11]

4. Given maps $X \to Y \to Z$ such that $Y \to Z$ and the composition $X \to Z$ are covering spaces. Show that $X \to Y$ is also a covering space if Z is locally path-connected, and in this case, it's normal if $X \to Z$ is.

Chapter 2: Homology

Motivation and Idea of Homology

1. Unsatisfying aspects of $\pi_1(X)$.

From $\pi_1(X)$ to $\pi_n(X)$, general homotopy group of homotopy classes for maps $S^n \to X$.

i) Since 2-dimensional CW complex is enough to have any group as the fundamental group, and so this invariant can not tell "dimension" (whatever it might be). It is certainly necessary to consider the higher dimension versions if one wants to understand all spaces.

ii) They are hard to compute. [Well, even $\pi_1(S^1)$ is not that easy to compute (or at least justify the result.)]
One of the difficulty is that you need to deal with the space of maps, which is "infinitely dimensional" (i.e. ridiculously huge) and sometimes has the topology of the original space hidden very behind.

iii) Sometimes, $\pi_1(X)$ is making life too complicated, for example, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. Clearly, a commutative version would be easier to handle.

2. Idea of Homology.

i) Still consider cycles like S^n in the space X, but make more direct use of the structure of the space, for example, CW complex structure.

ii) One gets for all dimensions in the same (and reasonable) way. There are quite a few equivalent ways in definition and the genuine computation is usually of combinatorics flavour. The basic algebraic construction is of general interest.

iii) We get commutative groups. In general, $H_1(X)$ is the commutative version of π_1. [This is not true for general dimensions, for example, S^n and $K(G, n)$.]

Section 2.1 Simplicial and Singular Homology

Δ-Complex

Recall: triangulation of Riemannian surface.
Vertices, edges and triangles. Euler number.
The idea of simplicial homology can be traced back to this.

To begin with, we need "triangle" for general dimension, which is "n-simplex", with all the ordering recorded. They can be used to build so called Δ-complex, which is more rigid than CW complex.

n**-simplex**: in $\mathbb{R}^m$ with $m \geqslant n$, the smallest convex set containing $n+1$ points (vertices) "in general position", i.e. not sitting in a hyperplane of dimension strictly smaller than n. [It is an object of dimension n.]

A **standard** n**-simplex**,
$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \sum_0^n t_i = 1, t_i \geqslant 0 \text{ for all } i\}$.
A general one is then,
$\{\sum_0^n t_i v_i \in \mathbb{R}^m \mid \sum_0^n t_i = 1, t_i \geqslant 0 \text{ for all } i\}$
where v_i's are vectors in $\mathbb{R}^m$ with $v_1 - v_0, \cdots, v_n - v_0$ linearly independent.
They are "linearly isomorphic", i.e. isomorphic under a linear injective map from $\mathbb{R}^{n+1} \to \mathbb{R}^m$.
$(t_0, \cdots, t_n)$ is called **barycentric coordinates** of $\sum_0^n t_i v_i$ in $\mathbb{R}^m$.

A **face** of the n-simplex is a subset which is the convex set containing some (maybe all but not none) of the vertices, which is a k-simplex by itself, with $k+1$ being the number of vertices involved.

The **ordering**: if we record the order or vertices from v_0 to v_n, the above general n-complex can be seen as n**-simplex of ordering** $[v_0, \cdots, v_n]$ **for the vertices**.
The ordering $[v_0, \cdots, v_n]$ **induces** ordering for the vertex set of any face.

Δ**-complex**: a collection of originally disjoint n-simplics (with ordering), glued together along (some) faces by order preserving linear homeomorphisms. [One can also use quotient space to describe this. Each simplex can be glued to itself, which is a feature more flexible than more traditionally defined simplicial complex.]

Remark: for a Δ-complex X,
1) there is orientation for each "face";
2) [Δ^n is diffeomorphic to D^n] there is clearly a characteristic

map $\sigma : \Delta^n \to X$ as for CW complex, only homeomorphic for interior (for example, gluing one triangle to get a cone), e^n, called open n-simplices.

Simplicial Homology

X is a Δ-complex.

$\Delta_n(X)$: the free abelian group generated by all open n-simplices (considered in X). [$\mathbb{Z}$-coeffient]

n-chain: element of $\Delta_n(X)$.

Warning: Δ_n and Δ^n mean totally different things for us.

Boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is defined as
$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|[v_0, \cdots, \widehat{v_i}, \cdots, v_n]$
on any open n-simplex e^n_α with characteristic map $\sigma_\alpha : D^n \to X$ [and extended to $\Delta_n(X)$].

Note: $\sigma_\alpha|[v_0, \cdots, \widehat{v_i}, \cdots, v_n]$ is indeed the characteristic map of an $(n-1)$-simplex of X.

Key Property: $\partial^2 = 0$, i.e. $0 = \partial_{n-1}\partial_n : \Delta_n \to \Delta_{n-1} \to \Delta_{n-2}$.

Proof: enough to notice $\sigma|[v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_n]$ appears twice with opposite signs.

This fits into the standard algebraic setting of chain complex and homology group.

Chain complex: $\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$, with abelian group C_n and homomorphism $\partial_n : C_n \to C_{n-1}$ for all n's, furthermore $\partial^2 = 0$, making sure $Im(\partial_{n+1}) \subset Ker(\partial_n)$.

Cycle: element of $Ker(\partial)$.

Boundary: element of $Im(\partial)$.

Homology group (for the chain complex): $H_n = Ker(\partial_n)/Im(\partial_{n+1})$ for all n's.

Homology class: element of homology group, i.e. equivalence class containing homologous (i.e. up to boundaries) cycles.

Let $C_n = \Delta_n(X)$ (0 for not proper n), **simplicial chain complex**, and we end up with **simplicial homology group** $H^\Delta_*(X)$. [To be more precise, $H^\Delta_*(X; \mathbb{Z})$.]

Examples:

i) $H^\Delta_n(S^1) = \mathbb{Z}$ for $n = 0, 1$ and 0 otherwise, using the Δ-complex structure of one vertex and one edge. [Not a traditional triangulation, but handy for computation.]

ii) $H^\Delta_n(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ for $n = 1$, $= \mathbb{Z}$ for $n = 0, 2$ and trivial for others.

Natural Questions:

1) X might have different Δ-complex structures. Would the choice matter? Would like the answer to be "NO".

2) If $X \simeq Y$ (or even $X = Y$, i.e. homeomorphic), would they have the same (simplicial homology)? Would also like the answer to be "NO" (for our purpose this semester, "homotopy invariant").

The answers are "NO". We shall use the equivalence of simplicial homology and another more abstractly defined (but more convenient theoretically) homology to justify them.

Also, for the first question, one can easily try the idea which involves refinement to give a more direct proof.

Good thing: $H_n^{\Delta}(X) = 0$ for n bigger than the "dimension" of X.

Remark: traditonally, simplicial complex is used to define simplicial homology. It is a special kind of Δ-complex such that, under the characteristic maps (i.e. considered in the topological space), any two different faces (simplex) can not have the same set of vertices. So the whole space can be recovered completely by combinatorial data, i.e. the set of vertices and subsets of it. Each characteristic map is homeomorphic from the closure to its image.

The choice made in [Hatcher] would benefit computation by reducing the number of simplices.

Example illustrating difference: for simplicial complex,

i) a simplex can not be glued to itself (the example above), i.e. each n-simplex has $n + 1$ vertices under the characteristic map;

ii) not just that, we need three vertices for S^1.

Singular Homology

1. Definition.

Singular n-simplex (of a space X): a map $\sigma : \Delta^n \to X$. [more general than simplicial simplices in Δ-complex]

$C_n(X)$: free abelian group generated by singular n-simplices. [infinite basis, finite summation, $\mathbb{Z}$-coefficient; of course, 0 for negative n]

n-chain: element of $C_n(X)$.

Boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined on any singular n-simplex σ (and hence extended to) as:

$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, \cdots, \widehat{v}_i, \cdots, v_n]$,

where each codimension one face of Δ^n is canonically considered as Δ^{n-1} as explained before.

In the same way as before, we can justify the **key property**, $\partial^2 = 0$, and have the notions of cycles and boundaries.

Hence we have the **singular homology group**,

$H_n(X) = Ker(\partial_n)/Im(\partial_{n+1})$. [$H_n(X;\mathbb{Z})$, to be more precise.]

By definition, easy to see *homeomorphic spaces have the same singular homology* as all the information collected is the same. [This can also be seen using the induced homomorphism later.]

Associated Δ-complex for X, $S(X)$: by induction,

i) start with X_0, a disjoint point set, with one vertex for each point in X;

ii) get one Δ^n for each $\sigma : \Delta^n \to X$ and glue it to X_{n-1} along identical faces $\Delta^{n-1} \to X$.

Clearly $H^\Delta(S(X)) = H_n(X)$, although $S(X)$ is huge (with infinite dimension, for example).

2. Basic Properties.

1) If $X = \cup_\alpha X_\alpha$ where X_α's are path-connected components, then $H_n(X) = \oplus_\alpha H_n(X_\alpha)$. [Each singular chain of X would have image in one of X_α.]

2) If X is path-connected, then $H_0(X) = \mathbb{Z}$. [0-chain is always cycle and any two 0-chains are homologous by 1-chain from a connecting path.]

3) If $X = \{p\}$, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbb{Z}$. [In this case, not so many chains.]

4) Reduced Homology.

For homology group $H_n(X)$, use the sequence

$\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to 0.$

For reduced homology group, $\widetilde{H}(X)$, use

$\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0,$

where the map $\epsilon : C_0(X) \to \mathbb{Z}$ is defined by "counting the coefficients" of 0-chains.

Claim: this sequence still has the key property, $\partial^2 = 0$.

Only need to check at $C_0(X)$ spot.

Relation between $H_*(X)$ and $\widetilde{H}_*(X)$:

$H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$: $[\sigma] \to ([\sigma - \epsilon(\sigma)x], \epsilon(\sigma))$ for any 0-chain σ.

$H_n(X) = \widetilde{H}_n(X)$ for $n > 0$.

5) For a path-connected X, $H_1(X)$ is the abelianization of $\pi_1(X)$.

Idea: to begin with, we see in $H_1(X)$, $[l] = \sum_i [l_i]$ if the loop l is the composition of paths l_i.

Then is an obvious surjective homomorphism $\Phi : \pi_1(X) \to H_1(X)$. [Being well-defined and surjective is justified by constructing proper singular 2-simplices. The fact above is handy.]

Then it's left to see the kernel is the subgroup for "commutativity". See detail in [Hatcher], Pages 166–167.

Homotopy Invariance

1. Induced Homomorphism.

Begin with a map $f : X \to Y$.
Directly induce $f_\sharp : C_n(X) \to C_n(Y)$ and also have $\partial \circ f_\sharp = f_\sharp \circ \partial$ (seen by definition). Also say $f_\sharp$ defines a **chain map** from singular chain complex of X to that of Y.

So we have a commutative diagram, and this justifies a induced homomorphism $f_* : H_n(X) \to H_n(Y)$ *(for all* n*).* [straightforward]

Sometimes, it's favourable to write it as $f_* : H_*(X) \to H_*(Y)$.

2. Basic Properties.

1) $f_* \circ g_* = (f \circ g)_* : H_*(X) \to H_*(Y) \to H_*(Z)$ for $g : X \to Y$ and $f : Y \to Z$.
[Clearly, $f_\sharp \circ g_\sharp = (f \circ g)_\sharp$, and so this is true.]

2) $Id_{X*} = Id : H_*(X) \to H_*(X)$.

These two are enough to show $X = Y$ (homeomorphic), then $H_*(X) = H_*(Y)$.

3. Homotopy Invariance.

Theorem: $f, g : X \to Y$ are homotopic, then
$f_* = g_* : H_*(X) \to H_*(Y)$,
and so $X \simeq Y$ implies $H_*(X) = H_*(Y)$.

Idea of Proof:

Construct a homomorphism $P : C_n(X) \to C_{n+1}(Y)$ by
$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times Id)|[v_0, \cdots, v_i, w_i, \cdots, w_n]$, where $\sigma \times Id : \Delta^n \times I \to X \times I \to Y$, $F : X \times I \to Y$ is the homotopy between f and g, and $\Delta^n \times I$ is broken into $n+1$ $(n+1)$-simplices with $\{v_i\}$ and $\{w_j\}$ being vertices for the bottom and top Δ^n respectively.
It is a **chain homotopy**: $P\partial + \partial P = g_\sharp - f_\sharp$. [direct check]

It takes pure algebraic consideration to see chain maps with chain homotopy between them induce isomorphism on homology.
[You would feel $\partial P = g_\sharp - f_\sharp$ is better, but less symmetry means it's less friendly in genuine construction.]

Remark: similar discussion works for reduced homology.

Exact Sequence and Excision

1. Exact Sequence and Quotient Space.

Exact sequence: sequence of homomorphisms $\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$ with $\alpha_n : A_n \to A_{n-1}$ satisfying $Ker(\alpha_n) = Im(\alpha_{n+1})$ for all n. [It'll give trivial homology information. Don't worry about the "endpoints".]

Short exact sequence: exact sequence like $0 \to A \to B \to C \to 0$. Clearly $A \to B$ injective and $B \to C$ surjective.

Recall: quotient space X/A.

Theorem A: if X is a space and A is a nonempty closed subspace that is a deformation retract of some neighbourhood in X (i.e. (X, A) is a "good pair"), then there is an exact sequence

$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A) \to \widetilde{H}_{n-1}(X) \to \cdots \to \widetilde{H}_0(X/A) \to 0$,

where $i_* : \widetilde{H}_*(A) \to \widetilde{H}_*(X)$ and $j_* : \widetilde{H}_*(X) \to \widetilde{H}_*(X/A)$ are the induced homomorphisms from the natural maps $i : A \to X$ and $j : X \to X/A$, and $\partial : \widetilde{H}_*(X/A) \to \widetilde{H}_{*-1}(A)$ is a homomorphism from a so-called "Snake Lemma Construction".

The proof is long, and so before diving into it, we see some applications first.

i) $\widetilde{H}_n(S^n) = \mathbb{Z}$ and $\widetilde{H}_i(S^n) = 0$ for $i \neq n$ for $n = 0, \cdots$.

Proof: [Not terrible computation for simplicial homology, but might look intimidating for singular homology.]

Take $(X, A) = (D^n, S^{n-1} = \partial D^n)$, and so $X/A = S^n$, good for $n = 1, \cdots$. [$n = 0$ is already known.]

D^n is contractible, and so $\widetilde{H}_*(D^n) = 0$.

So $\partial : \widetilde{H}_n(S^n) \to \widetilde{H}_{n-1}(S^{n-1})$ is isomorphic. So it's done by induction and the known S^0 case.

ii) For $n = 1, \cdots$, ∂D^n is not a retract of D^n, and so every map $f : D^n \to D^n$ has a fixed point. [$n = 2$ is proven before using fundamental group.]

Proof: suppose $r : D^n \to \partial D^n$ with $r \circ i = Id_{\partial D^n}$. Then look at the composition of induced homomorphisms and the result in i) to get contradiction.

2. Proof of **Theorem A**.

(I) Relative homology group.

$A \subset X$, $C_n(X, A) = C_n(X)/C_n(A)$. Clearly, ∂ for $C_*(X)$ and $C_*(A)$ would give a boundary map for $C_*(X, A)$, and this gives the**relative homology group**, $H_*(X, A)$ (or $H_*(X, A; \mathbb{Z})$, to be more precise).

Remark: different from $C_*(X/A)$ (and $H_*(X/A)$), but $\sigma : \Delta \to A$ is considered as 0 for $C_*(X, A)$, and as a constant map for $C_*(X/A)$.

Clearly, map between pairs induces homomorphism of relative homology.

Homotopy Invariance for Relative Homology: $f \simeq g : (X, A) \to (Y, B)$, then $f_* = g_* : H_*(X, A) \to H_*(Y, B)$.

[Proof is the same as that for the usual homology, using the same chain homotopy construction.]

(II) A general algebra fact: short exact sequence of chain complexes to long exact sequence of homology groups.

Stated in our context:

Result: the commutative diagram, $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$ is clearly a short exact sequence for each value of n. And it gives the following long exact sequence of homology groups:

$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots \to H_0(X, A) \to 0$,

where $H_*(A) \to H_*(X)$ and $H_*(X) \to H_*(X, A)$ are induces by chain maps, and *the boundary map* (Terrible name?) $H_*(X, A) \to H_{n-1}(A)$ is more complicated defined by chasing the diagram in a snaky way.

[There is also the version for reduced homology as is actually used to prove **Theorem A**:

$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to H_n(X, A) \to \widetilde{H}_{n-1}(A) \to \cdots \to H_0(X, A) \to 0$. For this, only need to add $0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$ at the bottom of the diagram.]

Proof:

Only define the non-obvious map. ["chasing the diagram"]

Remark: this is useful in computing homology.

Application: $H_*(X, x) = \widetilde{H}_*(X)$. [See $\partial : H_1(X, x) \to H_0(x) = \mathbb{Z}$ is trivial by the obvious injectivity of the next homomorphism.]

(III) **Excision Theorem**: $Z \subset A \subset X$ and $\overline{Z} \subset A^o$, then the inclusion $(X - Z, A - Z) \to (X, A)$ induces isomorphism on relative homology. Equivalently, $X = A^o \cup B^o$, then the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphism on relative homology.

[Take $B^c \subset A \subset X$ to see the equivalence of these two versions.]

Idea of proof:

Singular complexes with respect to a cover (the interiors forming an open cover) would give the same singular homology (through chain homotopy), i.e. the inclusion $C_*(A + B)/ \to C_*(X)/$ induces isomorphism on homology with the "inverse" being the Chain complex map ρ constructed using barycentric subdivision. This also works for the relative homology, i.e. the inclusion $C_*(A + B)/C_*(A) \to C_*(X)/C_*(A)$ induces isomorphism on homology.

[This is used again later when justifying Mayer-Vietoris sequence.]

It is obvious that $C_*(B)/C_*(B \cap A) \to C_*(A+B)/C_*(A)$ induces isomorphism on homology, and so the Excision Theorem is proven.

(IV) Finishing the Proof.

Result: for a good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphism on relative homology.

[Observe $H_*(X/A, A/A) = \widetilde{H}(X/A)$ to conclude the proof about long exact sequence involving quotient space.]

Idea of proof:

It involves the corresponding results (long exact sequence, etc.) for triples (X, V, A) (coming from $C_*(X, A)$), where V is a deformation retraction neighbourhood of A (using the "good pair" property).

$H_*(X, A) \to H_*(X, V)$ is isomorphic by long exact sequence and $H_*(V, A) = 0$,

$H_*(X - A, V - A) \to H_*(X, V)$ is isomorphic by excision.

Then consider the diagram consisting of the above ones in one row, the corresponding one after taking quotient by A in the other row, and the homomorphisms induced by quotient as the vertical ones.

Finally, notice $q_* : H_*(X - A, V - A) \to H_*(X/A - A/A, V/A - A/A)$ is isomorphic because this quotient is homeomorphic, resulting in all the vertical maps being isomorphic.

([Hatcher], bottom of Page 118 and Prop. 2.22 on Page 124)

3. Applications.

Result 1: Since, as a fact, CW pairs are always good pairs, if $X = A \cup B$ where A and B are subcomplices of X, then the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphism on relative homology.

Proof: pass to the quotient space and use $B/(A \cap B) = X/A$. [$H_*(X, A) = H_*(X/A, A/A) = \widetilde{H}_*(X/A)$ for a good pair (X, A) from before.]

Result 2: let $X = \vee_\alpha X_\alpha$. Inclusion $i_\alpha : X_\alpha \to X$ induces isomorphism $\oplus_\alpha i_{\alpha *} : \oplus_\alpha \widetilde{H}_*(X_\alpha) \to \widetilde{H}_*(X)$, if (X_α, x_α)'s are all good.

Proof: take $(Y, A) = (\sqcup_\alpha X_\alpha, \sqcup_\alpha \{x_\alpha\}) = \sqcup_\alpha (X_\alpha, x_\alpha)$ and also use $X = Y/A$.

Result 3: (dimension) open sets in $\mathbb{R}^n$ of different dimensions can not be homeomorphic.

Proof: $H_*(U, U \setminus \{x\}) = H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) = \widetilde{H}_{*-1}(\mathbb{R}^n \setminus \{x\})$ (by excision and then long exact sequence). [Reduced version is handy since $\widetilde{H}_*(\mathbb{R}^n) = 0$.]

Then use $\mathbb{R}^n \setminus \{x\} \simeq S^{n-1}$ and the homology for spheres to conclude.

4. Naturality, a Useful Digression.

(Space Map $\Rightarrow$) Chain Complex Homomorphism
$\Rightarrow$ Homology Homomorphism

Naturality: the diagram of homology induced by a commutative diagram of space map is commutative.

[No need to worry about the sometimes complicated construction for the induced homomorphisms all the time. Simply speaking, the construction from chain complex homomorphism to homology homomorphism is universal.]

[A map between pairs $f : (X, A) \to (Y, B)$ gives a commutative diagram of space map, and so induces commutative diagram between the long exact sequences.]

Equivalence Between Simplicial and Singular Homology

Result: for a Δ-complex, they are isomorphic, and so the possible choice of Δ structure won't be a factor.

Proof: [This also works for pairs with X being $(X, \varnothing)$.]
Use induction and check the affect of adding n-simplices.

The long exact sequence construction is also available for simplicial homology. So we have

$$H_k(X^n, X^{n-1}) \to H_{k-1}(X^{n-1}) \to H_{k-1}(X^n) \to H_{k-1}(X^n, X^{n-1}) \to H_{k-2}(X^{n-1})$$

for both H_*^Δ and H_*, with natural (so commutativity is OK) map $H_*^\Delta \to H_*$.
The second and fifth are isomorphic by induction.
The first and fourth are isomorphic by direct construction.([Hatcher] Page 129.)
[Basically, the inclusion map of each S^n generates $H_*(X^n, X^{n-1}) = \widetilde{H}_*(X^n/X^{n-1}) = \widetilde{H}_*(\vee S^n)$. More precisely, $H_n(S^n) = \mathbb{Z}(I - C_p)$ where $I : \Delta^n \to S^n$ maps $\partial\Delta^n$ to a point p and C_p is the constant map to p.]

Then one uses **Five Lemma** to conclude that the middle one is also isomorphic and get it done.

For a pair (X, A), we use long exact sequence for $H_*(X, A)$, the above case and Five Lemma to conclude.

Five Lemma: in a "5×2 commutative diagram", rows being exact and column homomorphisms 1, 2 , 4 and 5 being isomorphic implies the middle column is isomorphic.

Proof: diagram chasing, again.

Implication: for finite Δ-complex, singular homology is also finitely generated, as direct sums of $\mathbb{Z}$ and $\mathbb{Z}_m$.

Betti number: number of $\mathbb{Z}$ copies.
Torsion: all $\mathbb{Z}_m$ components, with *torsion coefficient* being m for $\mathbb{Z}_m$.

Suggested Problems

1. Justify the long exact sequence of homology involving relative homology.

2. Prove the Five Lemma.

3. Show that if A is a retract of X, then $H_*(A) \to H_*(X)$ induced by inclusion is injective.

[Hatcher, Section 2.1, Exercise 11]

4. Show that:

i) $H_0(X, A) = 0$ iff A meets each path component of X;

ii) $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path component of X contains at most one path component of A.

[Hatcher, Section 2.1, Exercise 16]

5. Give S^2 an explicit Δ-complex structure and compute the corresponding simplicial homology groups.

6. Let l_1, l_2 and l_3 be three paths in X with $l_1(0) = l_3(1)$, $l_1(1) = l_2(0)$ and $l_2(1) = l_3(0)$. Define the loop $l = l_3 \cdot l_2 \cdot l_1$ (based at $l_1(0)$ under our convention in this course). Show that l and $l_1+l_2+l_3$ are 1-cycles, and furthermore, $[l] = [l_1 + l_2 + l_3] \in H_1(X)$.

Section 2.2 Computations and Applications

Degree

1. Motivation: when discussing loop in S^1, i.e. map $S^1 \to S^1$, we essentially care for how many rounds the loop covers (which leads to $\pi_1(S^1)$). Here we generalize this idea for $S^n \to S^n$, using homology.

2. Definition and Basic Properties

For a map $f : S^n \to S^n$, the homomorphism $f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ is $\mathbb{Z} \to \mathbb{Z}$, and so $f_*(\alpha) = d \cdot \alpha$. The integer d is the **degree** of this map, $\deg f$.
[Need to be the same $\mathbb{Z}$ for well-definedness.]

Basic Properties: (maps $S^n \to S^n$)
a) $\deg Id = 1$.
b) $\deg f = 0$ if (but not only if) f is not surjective.
[factorize through $S^n \setminus \{p\}$ and "superficial" wraping of S^n]
c) $\deg(f \circ g) = \deg f \cdot \deg g$.
d) $\deg f = \deg g$ if $f \simeq g$. ("Only if" is true but deep. [Hopf])
e) If f is a reflection of S^n, then $\deg f = -1$. [simple Δ structure]
f) Antipodal map, $-Id$, has degree $(-1)^{n+1}$. [the composition of $(n+1)$ reflections]
g) f has no fixed points, then $\deg f = (-1)^{n+1}$.
[Interval between $f(x)$ and $-x$ won't go through origin. Then $\frac{(1-t)f(x)+t(-x)}{|(1-t)f(x)+t(-x)|}$ gives $f \simeq -Id$.]

3. More Results.

i) Existence of a nonzero continuous tangent vector field over S^n iff n is odd.

Proof:
For odd n, have $(-x_2, x_1, -x_4, x_3, \cdots, -x_{n+1}, x_n)$ at $(x_1, \cdots, x_n, x_{n+1})$
Suppose there is such a vector $v(x)$. $\cos t \cdot x + \sin t \cdot v(x)$ gives $Id \simeq -Id$. So $1 = (-1)^{n+1}$, and n odd.

ii) If n even, then $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n.

Proof: suppose group G acts freely on S^n.
Using degree, there is homomorphism $d : G \to \{\pm 1\}$ (as the action is homeomorphic).

Free action means "no-fixed-point" (for non-trivial element), and so degree is $(-1)^{n+1}$. For n even, all nontrival actions get mapped to -1, and the kernel of d is trivial. So $G \subset \mathbb{Z}_2$.

4. Local Degree.

$f : S^n \to S^n$. Suppose $f^{-1}(y) = \{x_i\}_{i\in F}$ for some finite set F.

Clearly, then $f : (U_i, U_i \setminus \{x_i\}) \to (V, V \setminus \{y\})$ induces $f_* : H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{y\})$.

Both groups are isomorphic to $H_n(S^n, S^n \setminus \{p\})$ by excision, and so $H_n(S^n)$ by long exact sequence.

So the induces f_* can be viewed as $\mathbb{Z} \to \mathbb{Z}$ (in a canonical way), and we have the integer called local degree of f at x_i, $\deg f|x_i$.

Result: $\deg f = \sum_i \deg f|x_i$ if there is such a y.

Proof: use $H_n(S^n, S^n \setminus f^{-1}(y)) = \oplus_i H_n(U_i, U_i \setminus \{x_i\})$, and the commutative diagram with

$f_* : H_n(S^n, S^n \setminus f^{-1}(y)) \to H_n(S^n, S^n \setminus \{y\})$, the summation of local degrees,

$f_* : H_n(S^n) \to H_n(S^n)$, the degree,

and the isomorphisms between $H_n(S^n)$, $H_n(S^n, S^n \setminus \{y\})$ and $H_n(V, V \setminus \{y\})$, together with the isomorphisms between $H_n(S^n)$, $H_n(S^n, S^n \setminus \{x_i\})$ and $H_n(U_i, U_i \setminus \{x_i\})$ for all i's.

[Hatcher, Page 136, Diagram.]

Cellular Homology

A very convenient homology for CW complex.

1. Facts for a CW complex X:

i) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$, and is free abelian for $k = n$, with basis corresponding to n-cells of X. [$\widetilde{H}_k(X^n/X^{n-1})$ and X^n/X^{n-1} is a wedge of n-spheres.]

ii) $H_k(X^n) = 0$ for $k > n$, and $H_k(X) = 0$ for $k > \dim X$. [long exact sequence for (X^n, X^{n-1})]

iii) The inclusion $X^n \to X$ induces isomorphism on homology for dimension $k < n$. [long exact sequence and $H_k(X^{n+1}, X^n) = \widetilde{H}_k(X^{n+1}/X^n) = 0$ for $k \leqslant n$, harder for CW complex of infinite dimension]

2. Cellular Homology.

$C_n = H_n(X^n, X^{n-1})$,
The boundary map for this chain complex is $d : C_n \to C_{n-1}$ is the composition of
$\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ and
$j_* : H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$
from long exact sequences for pairs (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}).
Clearly, $d^2 = 0$. [Hatcher, Page 139, Diagram]
It defines $H_*^{CW}(X)$.

Result: $H_*^{CW}(X) = H_*(X)$ for CW complex X.
[Chase through the diagram defining the cellular homology, applying the Facts discussed above.]

Remark: this is useful in computation, for example, homology of complex projective space.

The basis of the chains are cells of the corresponding dimension, and we also have a very intuitive understanding for the boundary map d.

Cellular Boundary Formula: $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ where $d_{\alpha\beta}$ is the degree of the map $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$ where $S^{n-1}_\beta = X^{n-1}/(X^{n-1} \setminus e^{n-1}_\beta)$.
[It can be viewed as the generalization of the boundary map for simplicial homology (linear and preserving orientation)]

[These two maps naturally corresponds to ∂ and j_* in the definition of d. One needs to view them properly. Do diagram chasing for a rigorous proof.]

Applications:

1) This can be used to construct space (CW complex) with desired homology information, for example, Moore Spaces ($H_n(X) = G$ and trivial otherwise).

2) Real projective space: one cell for each dimension with boundary map being $1 + (-1)^k$ in dimension k.

More precisely, $S^{k-1} \to \mathbb{RP}^{k-1} \to \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}$, where the left one is just the classic antipodal gluing to get $\mathbb{RP}^{k-1}$. One can then use local degree to calculate the degree. The pre-image of a proper point consists of two points. The local degree for one is the same as the identity map while the other one is the same as the antipodal map. That is how $1 + (-1)^k$ comes up.

3) Complex projective space: one cell for each even dimension, and so the boundary maps are all trivial.

3. Euler Characteristic.

$\chi(X) = \sum_n (-1)^n rank(H_n(X))$, in general.

Also, $\chi(X) = \sum_n (-1)^n c_n$, where c_n is the number of n-cells for any CW complex X.

Also, $\chi(X) = \sum_n (-1)^n C_n$, c_n is the number of n-simplices for Δ complex X. [triangulation for Riemann surface]

Proof: look at the chain complex used to define homology.

$\sum_n (-1)^n rank(H_n(X)) = \sum_n (-1)^n [rank(ker\partial_n) - rank(im\partial_{n+1})]$
$= \sum_n (-1)^n [rank(ker\partial_n) + rank(im\partial_n)] = \sum_n (-1)^n rank(C_n(X))$.

4. Split Exact Sequence

From short exact sequence $0 \to A \to B \to C \to 0$ (maps i and j), how do we have $B = A \oplus C$ compatible with the sequence?

Equivalent ways:

i) $p : B \to A$ with $p \circ i = Id_A$. [$B = i(A) \oplus Ker(p)$]

ii) $s : C \to B$ with $j \circ s = Id_C$. [$B = Ker(j) \oplus s(C)$]

Mayer-Vietoris Sequence

Idea: a long exact sequence of homology in the same spirit as Van Kampen Theorem for fundamental group.

Setting: $X = A^o \cup B^o$ for subspace A and B of X.
Short exact sequence for chain complexes:
$0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*(A + B) \to 0$
where $C_*(A + B)$ consists of chains with image in either A or B.
The map $C_*(A \cap B) \to C_*(A) \oplus C_*(B)$ is inclusion to $C_*(A)$ and $C_*(B)$ with a minus sign on one of them.
The map $C_*(A) \oplus C_*(B) \to C_*(A + B)$ is summation.
The inclusion $C_*(A + B) \hookrightarrow C_*(X)$ induces isomorphism on homology as mentioned in the proof of for Excision Theorem.

Thus, there is the long exact sequence for homology by the general algebraic construction, called Mayer-Vietoris Sequence:
$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0$,
and the same-looking version for reduced homology
$\cdots \to \widetilde{H}_n(A \cap B) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(A \cap B) \to \cdots \to \widetilde{H}_0(X) \to 0$.
There is also the relative version. [Hatcher, Page 152]

Applications:

a) Compute $H_*(S^n)$ using $S^n = D^n_N \cup D^n_S$, $D^n_N \cap D^n_S = S^{n-1}$ and doing induction by M-V sequence. This gives the inductive step $\widetilde{H}_k(S^n) = \widetilde{H}_{k-1}(S^{n-1})$.

b) Compute $H_*(T^2)$ using the decomposition
"$T^2 = D^2 \cup (S^1 \vee S^1)$, $D^2 \cap (S^1 \vee S^1) = (S^1)$"
and applying M-V sequence.
The non-trivial part of the long exact sequence is
$0 \to \widetilde{H}_2(T^2) \to \widetilde{H}_1(S^1) \to \widetilde{H}_1(S^1 \vee S^1) \to \widetilde{H}_1(T^2) \to 0$.
Need to figure out the homomorphism $\widetilde{H}_1(S^1) \to \widetilde{H}_1(S^1 \vee S^1)$.
The generator of $H_1(S^1)$ is mapped to $a + b + a^{-1} + b^{-1}$ where a and b are generators of $H_1(S^1 \vee S^1)$.
One can see in $H_1(S^1 \vee S^1)$, $a + a^{-1} = b + b^{-1} = [c_p] = 0$ where c_p is the constant map to the basepoint p.
So this map is trivial.

Homology with Coefficient

$C_*(X)$ is $\mathbb{Z}$ coefficient.

We can change the coefficient to another abelian group G and get $C_*(X;G)$ instead, typically $\mathbb{R}$ and $\mathbb{Z}_m$.

The whole machinery introduced before works in the same way and we get homology with G coefficient, $H_*(X;G)$, together with the whole package including long exact sequences.

Group homomorphism $G_1 \to G_2$ induces homology homomorphism $H_*(X;G_1) \to H_*(X;G_2)$, i.e. change of coefficients.

Example: $\mathbb{Z} \to G$.

Remark: Z coefficient homology contains the most complete homology information. [one version of "Universal Coefficient Theorem"]

Suggested Problems

1. Prove Brouwer Fixed Point Theorem for $f : D^n \to D^n$ (i.e. there has to be a fixed point) by applying degree theory to the map $F : S^n \to S^n$, defined using f, that sends both northern and southern hemisphere to the southern hemisphere.
[Hatcher, Section 2.2, Exercise 1]

2. Let $f : S^n \to S^n$ be a map of degree 0. Show that there exist x and y with $f(x) = x$ and $f(y) = -y$.
[Hatcher, Section 2.2, Exercise 3]

3. Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if $n > 0$.
[Hatcher, Section 2.2, Exercise 6]

4. SX is the suspension of X. Show by applying Mayer-Vietoris sequence that $\widetilde{H}_n(SX) = \widetilde{H}_{n-1}(X)$ for all n.
[Hatcher, Section 2.2, Exercise 32]

5. Let K be a knot, i.e. K homeomorphic to S^1, embedded in $\mathbb{R}^3$. Show that the only non-trivial (i.e. nonzero) $\widetilde{H}_n(\mathbb{R}^3 \setminus K)$ are $\mathbb{Z}$ for $n = 1, 2$.

Section 2.3 Formal Viewpoint

Idea: a lot of things we have done in defining and studying homology groups are quite universal, using general algebra tools. It's then natural to think about ways to make it axiomatic, i.e a priori without any explicit construction.

1. Axioms of Homology.

A **reduced homology theory** assigns to each nonempty CW complex X a sequence of abelian groups $\widetilde{h}_n(X)$ and to each map $f : X \to Y$ a sequence of homomorphisms $f_* : \widetilde{h}_*(X) \to \widetilde{h}_*(Y)$ such that

i) $(f \circ g)_* = f_* \circ g_*$, and $Id_* = Id$;

ii) $f \simeq g$ implies $f_* = g_*$;

iii) there are boundary homomorphisms $\partial : \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A)$ for each CW pair (X, A), fitting into the long exact sequence

$\cdots \to \widetilde{h}_n(A) \to \widetilde{h}_n(X) \to \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A) \to \cdots$.

Furthermore, the boundary map is natural in the sense that for any map of pairs $f : (X, A) \to (Y, B)$, the induced map $\bar{f} : X/A \to Y/B$ induce commutative square diagram involving f_*, $\bar{f}_*$ and ∂;

iv) for $X = \vee_\alpha X_\alpha$, the inclusions i_α induce isomorphism $\oplus_\alpha i_{\alpha *} : \oplus_\alpha \widetilde{h}_n(X_\alpha) \to \widetilde{h}_n(X)$ for all n.

Remark: we clearly have all these for the homology defined earlier by explicit constructions.

All other properties can be deduced from them.

Coefficient: $\widetilde{h}_0(S^0)$.

Also have Mayer-Vietoris Sequence.

2. Categories and Functors

Idea: going one step further in the world of abstract nonsense.

Only introduce some examples.

Category: topological spaces and (continuous) maps.

[i.e. the object under study]

Functor: induced homology groups and homomorphisms (covariant).

[i.e. the machinery used to study the object]